

# Poincaré duality, Hilbert complexes and geometric applications

Francesco Bei

## Abstract

Let  $(M, g)$  an open and oriented riemannian manifold. The aim of this paper is to study some properties of the two following sequences of  $L^2$  cohomology groups:

1.  $H_{2,m \rightarrow M}^i(M, g)$  defined as the image  $\text{im}(H_{2,min}^i(M, g) \rightarrow H_{2,max}^i(M, g))$
2.  $\overline{H}_{2,m \rightarrow M}^i(M, g)$  defined as  $\text{im}(\overline{H}_{2,min}^i(M, g) \rightarrow \overline{H}_{2,max}^i(M, g))$ .

We show, under certain hypothesis, that the first sequence is the cohomology of a suitable Hilbert complex which contains the minimal one and is contained in the maximal one. We also show that when the second sequence is finite dimensional then Poincaré duality holds for it and that, in the same assumptions, when  $\dim(M) \equiv 0 \pmod{4}$  we can use it to define a  $L^2$  signature on  $M$ . Moreover we show several applications to the intersection cohomology of compact smoothly stratified pseudomanifolds and we get some results about the Friedrichs extension  $\Delta_i^{\mathcal{F}}$  of  $\Delta_i$ .

## Introduction

The study of singular spaces from a geometric differential point of view leads naturally to consider (and to study) open differentiable manifolds with incomplete riemannian metric. As examples we can think to the great variety of papers about the relationships between the  $L^2$  Hodge and de Rham cohomology associated to some incomplete riemannian metric and the intersection cohomology of the whole singular space relative to some perversity associated to the metric. Among them, and the following list is just a brief reminder without any goals of completeness, we have the famous paper of Cheeger [8], the papers of Nagase [22], [23], the paper of Hunsicker and Hunsicker and Mazzeo [14] and [15] and the paper of Saper [24]. Nevertheless, as it is well known, when  $M$  is an open manifold and  $g$  is an incomplete riemannian metric on  $M$  then the de Rham differential  $d_i : L^2\Omega^i(M, g) \rightarrow L^2\Omega^{i+1}(M, g)$  could have many closed extensions when we look at it as an unbounded operator defined over the smooth forms with compact support. This implies that there exists several way to turn the complex  $(\Omega_c^i(M), d_i)$  into an Hilbert complex and perhaps the most natural ones are to consider  $(L^2\Omega^i(M, g), d_{max,i})$  and  $(L^2\Omega^i(M, g), d_{min,i})$  where  $d_{max,i} : L^2\Omega^i(M, g) \rightarrow L^2\Omega^{i+1}(M, g)$  is defined in the distributional sense and  $d_{min,i} : L^2\Omega^i(M, g) \rightarrow L^2\Omega^{i+1}(M, g)$  is defined as the closure, under the graph norm, of  $d_i : \Omega_c^i(M) \rightarrow \Omega_c^{i+1}(M)$ . So a natural question is:

- if this two Hilbert complexes have finite dimensional  $L^2$  cohomology groups or finite dimensional reduced  $L^2$  cohomology groups, does Poincaré duality hold for them?

As it is well know the answer is usually no in both cases.

Anyway, from the pair of Hilbert complexes  $(L^2\Omega^i(M, g), d_{max/min,i})$  we can get other sequences of  $L^2$  cohomology groups defined in the following way:

$$H_{2,m \rightarrow M}^i(M, g), \text{ defined as the image } H_{2,min}^i(M, g) \longrightarrow H_{2,max}^i(M, g) \quad (1)$$

$$\overline{H}_{2,m \rightarrow M}^i(M, g), \text{ defined as the image } \overline{H}_{2,min}^i(M, g) \longrightarrow \overline{H}_{2,max}^i(M, g). \quad (2)$$

where in (1), as well as in (2), the map is the map induced in cohomology (reduced cohomology) by the natural inclusion of complexes  $(L^2\Omega^i(M, g), d_{min,i}) \subseteq (L^2\Omega^i(M, g), d_{max,i})$ .

At this point we can summarize the goal of this paper in the following way:

- Investigate the properties of the groups,  $H_{2,m \rightarrow M}^i(M, g)$ ,  $\overline{H}_{2,m \rightarrow M}^i(M, g)$   $i = 0, \dots, \dim M$ . In particular show that, under suitable hypothesis, the first sequence is the cohomology of a suitable complex which contains the minimal one and is contained in the maximal one and that, when the sequence defined in 2 is finite dimensional, then Poincaré duality holds for it. In particular, joining this two properties, we will show that when  $(L^2\Omega^i(M, g), d_{\max/\min, i})$  is a Fredholm complex then  $H_{2,m \rightarrow M}^i(M, g)$  is the cohomology of a Fredholm complex for which Poincaré duality holds. Then we show that, when  $\dim(M) = 4n$ , we can use again the sequence 2 to define a  $L^2$  signature on  $M$  and to get the existence of a topological signature on  $M$ . Moreover we show several applications to stratified pseudomanifolds and we get a topological obstruction to the existence of a riemannian metric (complete or incomplete) with finite  $L^2$  cohomology (reduced and unreduced). Finally we show some applications to  $\Delta_i^{\mathcal{F}}$ , the Friedrichs extension of  $\Delta_i$ ; in particular we prove that when  $(L^2\Omega^i(M, g), d_{\max/\min, i})$  are Fredholm complexes, then  $\Delta_i^{\mathcal{F}}$  is a Fredholm operator for each  $i$ . This last result applies, for example, when  $M$  is the regular part of a compact and smoothly stratified pseudomanifold with a Thom-Mather stratification.

The paper is structured in the following way:

in the first section we introduce the notion of Hilbert complexes; we generalize to this abstract framework the properties of the pair  $(L^2\Omega^i(M, g), d_{\min, i}) \subset (L^2\Omega^i(M, g), d_{\max, i})$ . In particular in definition 2 we introduce the notion of complementary Hilbert complexes, that is a pair of Hilbert complexes  $(H_j, D_j) \subseteq (H_j, L_j)$  such that there exist an isometry  $\phi_i : H_i \rightarrow H_{n-i}$  which satisfies  $\phi_i(\mathcal{D}(D_i)) = \mathcal{D}(L_{n-i-1}^*)$  and  $L_{n-i-1}^* \circ \phi_i = C_i(\phi_{i+1} \circ D_i)$  on  $\mathcal{D}(D_i)$ , where  $L_{n-i-1}^* : H_{n-i} \rightarrow H_{n-i-1}$  is the adjoint of  $L_{n-i-1} : H_{n-i-1} \rightarrow H_{n-i}$  and  $C_i$  is a constant which depends only on  $i$ . Then we prove these two theorems:

**Theorem 1.** *Let  $(H_j, D_j) \subseteq (H_j, L_j)$  a pair of complementary Hilbert complexes. Let  $i_{r,j}^*$  be the map induced by the inclusion of complexes between the reduced cohomology groups. Suppose that for each  $j$*

$$\text{im}(\overline{H}^j(H_*, D_*) \xrightarrow{i_{r,j}^*} \overline{H}^j(H_*, L_*)) \quad (3)$$

*is finite dimensional. Then*

$$\text{im}(\overline{H}^j(H_*, D_*) \xrightarrow{i_{r,j}^*} \overline{H}^j(H_*, L_*)), \quad j = 0, \dots, n \quad (4)$$

*is a finite sequences of finite dimensional vector spaces with Poicaré duality.*

The second theorem describe an abstract framework in which the groups  $\text{im}(H^i(H_*, D_*) \rightarrow H^i(H_*, L_*))$  are effectively the cohomology groups of an Hilbert complex which is intermediate between  $(H_j, D_j)$  and  $(H_j, L_j)$ :

**Theorem 2.** *Let  $(H_j, D_j) \subseteq (H_j, L_j)$   $j = 0, \dots, n$  a pair of Hilbert complexes. Suppose that for each  $j$   $\text{ran}(D_j)$  is closed in  $H_{j+1}$ . Then there exists a third Hilbert complex  $(H_j, P_j)$  such that*

1.  $(H_j, D_j) \subseteq (H_j, P_j) \subseteq (H_j, L_j)$
2.  $H^i(H_*, P_*) = \text{im}(H^i(H_*, D_*) \rightarrow H^i(H_*, L_*))$ .

*Moreover if  $(H_j, D_j) \subseteq (H_j, L_j)$  are complementary and  $(H_j, D_j)$  or equivalently  $(H_j, L_j)$  is Fredholm then  $(H_j, P_j)$  is a Fredholm complex with Poincaré duality.*

In the second section we specialize the situation to the pair of complementary Hilbert complexes that are natural in riemannian geometry; our main results are the following two theorems which are a consequence of the two previous results:

**Theorem 3.** *Let  $(M, g)$  an open, oriented and incomplete riemannian manifold of dimension  $m$ . Then the complexes*

$$(L^2\Omega^i(M, g), d_{\max, i}) \text{ and } (L^2\Omega^i(M, g), d_{\min, i})$$

are a pair of complementary Hilbert complexes.

In particular if  $\text{im}(\overline{H}_{2,\min}^i(M, g) \xrightarrow{i_{r,i}^*} \overline{H}_{2,\max}^i(M, g))$  is finite dimensional for each  $i$  then

$$\text{im}(\overline{H}_{2,\min}^i(M, g) \xrightarrow{i_{r,i}^*} \overline{H}_{2,\max}^i(M, g))$$

is a finite sequence of finite dimensional vector spaces with Poincaré duality.

In particular this last result is a geometric application of theorem 2:

**Theorem 4.** *Let  $(M, g)$  an open, oriented and incomplete riemannian manifold of dimension  $n$ . Suppose that for each  $i$   $\text{ran}(d_{\min,i})$  is closed in  $L^2\Omega^{i+1}(M, g)$ . Then there exists a Hilbert complex  $(L^2\Omega^i(M, g), d_{\mathfrak{m},i})$  such that for each  $i = 0, \dots, n$*

$$\mathcal{D}(d_{\min,i}) \subset \mathcal{D}(d_{\mathfrak{m},i}) \subset \mathcal{D}(d_{\max,i}),$$

$d_{\max,i}$  is an extension of  $d_{\mathfrak{m},i}$  which is an extension of  $d_{\min,i}$  and

$$H_{2,\mathfrak{m}}^i(M, g) = \text{im}(H_{2,\min}^i(M, g) \xrightarrow{i_i^*} H_{2,\max}^i(M, g))$$

where  $H_{2,\mathfrak{m}}^i(M, g)$  is the cohomology of the Hilbert complex  $(L^2\Omega^i(M, g), d_{\mathfrak{m},i})$ . Finally, if  $(L^2\Omega^i(M, g), d_{\max,i})$  or equivalently  $(L^2\Omega^i(M, g), d_{\min,i})$  is Fredholm, then  $(L^2\Omega^i(M, g), d_{\mathfrak{m},i})$  is a Fredholm complex with Poincaré duality.

From the previous theorem we get as corollary that under certain conditions it is possible to construct a self-adjoint extension of  $\Delta_i : \Omega_c^i(M) \rightarrow \Omega_c^i(M)$ , the Laplacian acting on the space of smooth compactly supported  $i$ -forms, such that it is a Fredholm operator with nullspace isomorphic to  $\text{im}(H_{2,\min}^i(M, g) \xrightarrow{i_i^*} H_{2,\max}^i(M, g))$ .

**Corollary 1.** *In the same assumptions of theorem 12; Let  $\Delta_i : \Omega_c^i(M) \rightarrow \Omega_c^i(M)$  be the Laplacian acting on the space of smooth compactly supported forms. Then there exists a self-adjoint extension  $\Delta_{\mathfrak{m},i} : L^2\Omega^i(M, g) \rightarrow L^2\Omega^i(M, g)$  with closed range such that*

$$\text{Ker}(\Delta_{\mathfrak{m},i}) \cong \text{im}(H_{2,\min}^i(M, g) \xrightarrow{i_i^*} H_{2,\max}^i(M, g)).$$

Moreover, if  $(L^2\Omega^i(M, g), d_{\max,i})$  or equivalently  $(L^2\Omega^i(M, g), d_{\min,i})$  is Fredholm, then  $\Delta_{\mathfrak{m},i}$  is a Fredholm operator on its domain endowed with the graph norm.

In rest of the chapter we state several applications of these results; in particular we show that there be topological obstructions to the existence of a riemannian metric with some analytic properties, see corollary 8.

Finally, when  $(M, g)$  is an open, oriented and incomplete riemannian manifold of dimension  $4n$  such that  $\text{im}(\overline{H}_{2,\min}^{2n}(M, g) \rightarrow \overline{H}_{2,\max}^{2n}(M, g))$  is finite dimensional, we show that it is possible to define a  $L^2$  **signature** on  $M$  and that this implies also the existence of a **topological signature** on  $M$ ; see definition 6 and prop. 18.

The third section is devoted to the applications of the previous results to compact smoothly stratified pseudomanifold with a Thom-Mather stratification; after recalled the  $L^2$  Hodge and de Rham theorem stated in [4], we get some consequences for the intersection cohomology groups associated to some general perversity in the sense of Friedman; see proposition 21, corollaries 10, 11, 12 and 16. In particular we have the following:

**Theorem 5.** *Let  $X$  be a compact smoothly stratified pseudomanifold of dimension  $n$  with a Thom-Mather stratification. Let  $g$  be a quasi edge metric with weights on  $\text{reg}(X)$ , see def. 8. Then we have the following results:*

$$\text{Ker}(\Delta_{\mathfrak{m},i}) \cong \text{im}(I^{q_g} H^i(X, \mathcal{R}_0) \rightarrow I^{p_g} H^i(X, \mathcal{R}_0)) \quad (5)$$

$$\text{ind}((d_{\mathfrak{m}} + d_{\mathfrak{m}}^*)_{ev}) = I^{p_g \rightarrow q_g} \chi(X, \mathcal{R}_0) \quad (6)$$

where  $I^{p_g \rightarrow q_g} \chi(X, \mathcal{R}_0) = \sum_i (-1)^i \text{im}(I^{q_g} H^i(X, \mathcal{R}_0) \rightarrow I^{p_g} H^i(X, \mathcal{R}_0))$  and  $(d_{\mathfrak{m}} + d_{\mathfrak{m}}^*)_{ev}$  is the extension of

$$d + \delta : \bigoplus_i \Omega_c^{2i}(M) \rightarrow \bigoplus_i \Omega_c^{2i+1}(M) \text{ defined by } (d_{\mathfrak{m}} + d_{\mathfrak{m}}^*)_{ev}|_{L^2 \Omega^{2i}(M, g)} := d_{\mathfrak{m}, 2i} + d_{\mathfrak{m}, 2i-1}^*$$

which is a Fredholm operator on its domain endowed with the graph norm.

Moreover we remark that in this framework the  $L^2$  signature introduced in the previous section in a more general context has a topological meaning because it coincides with the **perverse signature** introduced by Friedman and Hunsicker in [13], that is

$$\sigma_2(\text{reg}(X), g) = \sigma_{q_g \rightarrow p_g}(X).$$

Finally in the last section we show some applications to  $\Delta_i^{\mathcal{F}}$ , the Friedrichs extension of  $\Delta_i$ . Our main result is:

**Theorem 6.** *Let  $(M, g)$  be an open, oriented and incomplete riemannian manifold such that  $(L^2 \Omega^i(M, g), d_{\max, i})$ , or equivalently  $(L^2 \Omega^i(M, g), d_{\min, i})$ , is a Fredholm complex. Then for each  $i$ ,  $\Delta_i^{\mathcal{F}}$ , the Friedrichs extension of  $\Delta_i : \Omega_c^i(M) \rightarrow \Omega_c^i(M)$ , is a Fredholm operator on its domain endowed with the graph norm. Moreover  $\text{Ker}(\Delta_i^{\mathcal{F}}) = \text{Ker}(\Delta_{\min, i})$  and  $\text{ran}(\Delta_i^{\mathcal{F}}) = \Delta_{\max, i}$ .*

As a particular case of the previous theorem we have the following corollary:

**Corollary 2.** *Let  $X$  be a compact smoothly and oriented stratified pseudomanifold of dimension  $n$  with a Thom Mather stratification. Let  $g$  be a quasi-edge metric with weights on  $\text{reg}(X)$ . Then on  $L^2 \Omega^i(\text{reg}(X), g)$ , for each  $i = 0, \dots, n$ ,  $\Delta_i^{\mathcal{F}}$  is a Fredholm operator; moreover  $\text{ran}(\Delta_i^{\mathcal{F}}) = \text{ran}(\Delta_{\max, i})$  and  $\text{Ker}(\Delta_i^{\mathcal{F}}) = \text{Ker}(\Delta_{\min, i}) = \mathcal{H}_{\min}^i(M, g)$ .*

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## 1 Hilbert Complexes

We start the section recalling the notion of Hilbert complex and its main properties. For a complete development of the subject we refer to [5].

**Definition 1.** *A Hilbert complex is a complex,  $(H_*, D_*)$  of the form:*

$$0 \rightarrow H_0 \xrightarrow{D_0} H_1 \xrightarrow{D_1} H_2 \xrightarrow{D_2} \dots \xrightarrow{D_{n-1}} H_n \rightarrow 0, \quad (7)$$

where each  $H_i$  is a separable Hilbert space and each map  $D_i$  is a closed operator called the differential such that:

1.  $\mathcal{D}(D_i)$ , the domain of  $D_i$ , is dense in  $H_i$ .
2.  $\text{ran}(D_i) \subset \mathcal{D}(D_{i+1})$ .
3.  $D_{i+1} \circ D_i = 0$  for all  $i$ .

The cohomology groups of the complex are  $H^i(H_*, D_*) := \text{Ker}(D_i) / \text{ran}(D_{i-1})$ . If the groups  $H^i(H_*, D_*)$  are all finite dimensional we say that it is a *Fredholm complex*.

Given a Hilbert complex there is a dual Hilbert complex

$$0 \leftarrow H_0 \xleftarrow{D_0^*} H_1 \xleftarrow{D_1^*} H_2 \xleftarrow{D_2^*} \dots \xleftarrow{D_{n-1}^*} H_n \leftarrow 0, \quad (8)$$

defined using  $D_i^* : H_{i+1} \rightarrow H_i$ , the Hilbert space adjoints of the differentials  $D_i : H_i \rightarrow H_{i+1}$ . The cohomology groups of  $(H_j, (D_j)^*)$ , the dual Hilbert complex, are

$$H^i(H_j, (D_j)^*) := \text{Ker}(D_{n-i-1}^*) / \text{ran}(D_{n-i}^*).$$

For all  $i$  there is also a laplacian  $\Delta_i = D_i^* D_i + D_{i-1} D_{i-1}^*$  which is a self-adjoint operator on  $H_i$  with domain

$$\mathcal{D}(\Delta_i) = \{v \in \mathcal{D}(D_i) \cap \mathcal{D}(D_{i-1}^*) : D_i v \in \mathcal{D}(D_i^*), D_{i-1}^* v \in \mathcal{D}(D_{i-1})\} \quad (9)$$

and nullspace:

$$\mathcal{H}^i(H_*, D_*) := \text{ker}(\Delta_i) = \text{Ker}(D_i) \cap \text{Ker}(D_{i-1}^*). \quad (10)$$

The following propositions are standard results for these complexes. The first result is a weak Kodaira decomposition:

**Proposition 1.** *[[5], Lemma 2.1] Let  $(H_i, D_i)$  be a Hilbert complex and  $(H_i, (D_i)^*)$  its dual complex, then:*

$$H_i = \mathcal{H}^i \oplus \overline{\text{ran}(D_{i-1})} \oplus \overline{\text{ran}(D_i^*)}.$$

The reduced cohomology groups of the complex are:

$$\overline{H}^i(H_*, D_*) := \text{Ker}(D_i) / (\overline{\text{ran}(D_{i-1})}).$$

By the above proposition there is a pair of weak de Rham isomorphism theorems:

$$\begin{cases} \mathcal{H}^i(H_*, D_*) \cong \overline{H}^i(H_*, D_*) \\ \mathcal{H}^i(H_*, D_*) \cong \overline{H}^{n-i}(H_*, (D_*)^*) \end{cases} \quad (11)$$

where in the second case we mean the cohomology of the dual Hilbert complex.

The complex  $(H_*, D_*)$  is said *weak Fredholm* if  $\mathcal{H}_i(H_*, D_*)$  is finite dimensional for each  $i$ . By the next propositions it follows immediately that each Fredholm complex is a weak Fredholm complex.

**Proposition 2.** *[[5], corollary 2.5] If the cohomology of a Hilbert complex  $(H_*, D_*)$  is finite dimensional then, for all  $i$ ,  $\text{ran}(D_{i-1})$  is closed and  $H^i(H_*, D_*) \cong \mathcal{H}^i(H_*, D_*)$ .*

**Proposition 3** ([5], corollary 2.6). *A Hilbert complex  $(H_j, D_j)$ ,  $j = 0, \dots, n$  is a Fredholm complex (weak Fredholm) if and only if its dual complex,  $(H_j, D_j^*)$ , is Fredholm (weak Fredholm). If it is Fredholm then*

$$\mathcal{H}_i(H_j, D_j) \cong H_i(H_j, D_j) \cong H_{n-i}(H_j, (D_j)^*) \cong \mathcal{H}_{n-i}(H_j, (D_j)^*). \quad (12)$$

Analogously in the weak Fredholm case we have:

$$\mathcal{H}_i(H_j, D_j) \cong \overline{H}_i(H_j, D_j) \cong \overline{H}_{n-i}(H_j, (D_j)^*) \cong \mathcal{H}_{n-i}(H_j, (D_j)^*). \quad (13)$$

Now we recall another result which shows that it is possible to compute the cohomology groups of an Hilbert complex using a core subcomplex

$$\mathcal{D}^\infty(H_i) \subset H_i.$$

For all  $i$  we define  $\mathcal{D}^\infty(H_i)$  as consisting of all elements  $\eta$  that are in the domain of  $\Delta_i^l$  for all  $l \geq 0$ .

**Proposition 4** ([5], Theorem 2.12). *The complex  $(\mathcal{D}^\infty(H_i), D_i)$  is a subcomplex quasi-isomorphic to the complex  $(H_i, D_i)$ .*

Given a pair of Hilbert complexes  $(H_j, D_j)$  and  $(H_j, D'_j)$  we will write  $(H_j, D_j) \subseteq (H_j, D'_j)$  if for each  $j$  one of the two following properties is satisfied:

1.  $D'_j : H_i \rightarrow H_{j+1}$  is equal to  $D_j : H_j \rightarrow H_{j+1}$
2.  $D'_j : H_j \rightarrow H_{j+1}$  is a proper closed extension of  $D_j : H_i \rightarrow H_{j+1}$

We will write  $(H_j, D_j) \subset (H_j, D'_j)$  when the second of the above properties is satisfied. For each  $j$  let  $i_j : \mathcal{D}(D_j) \rightarrow \mathcal{D}(L_j)$  denote the natural inclusion of the domain of  $D_j$  into the domain of  $L_j$ . Obviously  $i_j$  induces a maps between  $H^j(H_*, D_*)$  and  $H^j(H_*, L_*)$  and between  $\overline{H}^j(H_*, D_*)$  and  $\overline{H}^j(H_*, L_*)$ . We will label the first as

$$i_j^* : H^j(H_*, D_*) \rightarrow H^j(H_*, L_*) \quad (14)$$

and the second as

$$i_{r,j}^* : \overline{H}^j(H_*, D_*) \rightarrow \overline{H}^j(H_*, L_*) \quad (15)$$

Consider again a pair of Hilbert complexes  $(H_i, D_i)$  and  $(H_i, L_i)$  with  $i = 0, \dots, n$ .

**Definition 2.** The pair  $(H_i, D_i)$  and  $(H_i, L_i)$  is said to be **related** if the following property is satisfied

- for each  $i$  there exist a linear, continuous and bijective map  $\phi_i : H_i \rightarrow H_{n-i}$  such that  $\phi_i(\mathcal{D}(D_i)) = \mathcal{D}(L_{n-i-1}^*)$  and  $L_{n-i-1}^* \circ \phi_i = C_i(\phi_{i+1} \circ D_i)$  on  $\mathcal{D}(D_i)$  where  $L_{n-i-1}^* : H_{n-i} \rightarrow H_{n-i-1}$  is the adjoint of  $L_{n-i-1} : H_{n-i-1} \rightarrow H_{n-i}$  and  $C_i \neq 0$  is a constant which depends only on  $i$ .

Furthermore we call the maps  $\phi_i$  link maps.

- We call the complexes **complementary** if each  $\phi_i$  is an isometry between  $H_i$  and  $H_{n-i}$ .

We have the following propositions:

**Proposition 5.** Let  $(H_i, D_i)$  and  $(H_i, L_i)$  be related Hilbert complexes. Then

1. Also  $(H_i, L_i)$  and  $(H_i, D_i)$  are related Hilbert complexes. Moreover if  $\{\phi_i\}$  are the link maps which make  $(H_i, D_i)$  and  $(H_i, L_i)$  related then  $\{\phi_i^*\}$ , the family of respective adjoint maps, are the link maps which make  $(H_i, L_i)$  and  $(H_i, D_i)$  related.
2. The complexes  $(H_i, D_i)$  and  $(H_i, L_i^*)$  have isomorphic cohomology groups and isomorphic reduced cohomology groups. In the same way the complexes  $(H_i, L_i)$  and  $(H_i, D_i^*)$  have isomorphic cohomology groups and isomorphic reduced cohomology groups.
3. The following isomorphisms hold:  $\mathcal{H}^j(H_i, D_i) \cong \mathcal{H}^{n-j}(H_i, L_i)$ ,  $\overline{\mathcal{H}}^j(H_i, D_i) \cong \overline{\mathcal{H}}^{n-j}(H_i, L_i)$ .
4. If the complexes  $(H_i, D_i)$  and  $(H_i, L_i^*)$  are complementary then each  $\phi_j$  induces an isomorphism between  $\mathcal{H}^j(H_i, D_i)$  and  $\mathcal{H}^{n-j}(H_i, L_i)$ .

*Proof.* By definition 2 we know that  $\phi_i^* : H_{n-i} \rightarrow H_i$ , the adjoint of  $\phi_i : H_i \rightarrow H_{n-i}$ , is a family of linear continuous and bijective maps. In this way if we look at  $L_{n-i-1}^* \circ \phi_i$  as an unbounded linear map between  $H_i$  and  $H_{n-i-1}$  with domain  $\mathcal{D}(L_{n-i-1}^* \circ \phi_i) = \phi_i^{-1}(\mathcal{D}(L_{n-i-1}^*))$  we have that  $(L_{n-i-1}^* \circ \phi_i)^* = \phi_i^* \circ L_{n-i-1}$  that is the adjoint of  $L_{n-i-1}^* \circ \phi_i$  is  $\phi_i^* \circ L_{n-i-1}$  with  $\mathcal{D}(\phi_i^* \circ L_{n-i-1}) = \mathcal{D}(L_{n-i-1})$ .

In the same way we have  $(\phi_{i+1} \circ D_i)^* = (D_i^* \circ \phi_{i+1}^*)$  where  $\mathcal{D}(\phi_{i+1} \circ D_i) = \mathcal{D}(D_i)$  and  $\mathcal{D}(D_i^* \circ \phi_{i+1}^*) = (\phi_{i+1}^*)^{-1}(\mathcal{D}(D_i^*))$ . In this way it follows that  $C_i(D_i^* \circ \phi_{i+1}^*) = \phi_i^* \circ L_{n-i-1}$  on  $\mathcal{D}(L_{n-i-1})$  for each  $i$  and clearly  $\phi_{i+1}^*(\mathcal{D}(L_{n-i-1})) = \mathcal{D}(D_i^*)$ . So we can conclude that the complexes  $(H_i, L_i)$  and  $(H_i, D_i)$  are related with  $\{\phi_i^*\}$  as link maps.

The second property is an immediate consequences of definition 2 and the first point of the proposition. Now if we compose the isomorphisms of the second point with the isomorphisms of (11) we can get the isomorphisms of the third point. Finally if each  $\phi_i$  is an isometry then  $\phi_i^* = \phi_i^{-1}$ . By definition 2 we know that  $\phi_i$  induces an isomorphism between  $\text{Ker}(D_i)$  and  $\text{Ker}(L_{n-i-1}^*)$ . In the same way by the first point of the proposition we know that  $\phi_i^*$  induces an isomorphism between  $\text{Ker}(L_{n-i})$  and  $\text{Ker}(D_{i-1}^*)$ . But now we know that  $\phi_i^* = \phi_i^{-1}$  and so we can conclude that for each  $i$   $\phi_i$  induces an isomorphism between  $\text{Ker}(D_i) \cap \text{Ker}(D_{i-1}^*)$  and  $\text{Ker}(L_{n-i}) \cap \text{Ker}(L_{n-i-1}^*)$ , that is an isomorphism between  $\mathcal{H}^i(H_*, D_*)$  and  $\mathcal{H}^{n-i}(H_*, L_*)$ .  $\square$

**Proposition 6.** *Let  $(H_i, D_i)$ ,  $i = 0, \dots, n$  a Hilbert complex and suppose that for each  $i$  there exists  $\phi_i : H_i \rightarrow H_{n-i}$  that is linear, continuous and bijective. Then there exist a Hilbert complex  $(H_i, L_i)$  such that the complexes  $(H_i, D_i)$  and  $(H_i, L_i)$  are related with  $\{\phi_i\}$  as link maps. Moreover if each  $\phi_j$  is an isometry then the complexes  $(H_i, D_i)$  and  $(H_i, L_i)$  are complementary with  $\{\phi_i\}$  as link maps.*

*Proof.* Consider the following complexes  $(H_i, L_i)$  where each  $L_i$  is the adjoint of the closed and densely defined operator  $(\phi_{n-i} \circ D_{n-i-1} \circ \phi_{n-i-1}^{-1}) : H_{i+1} \rightarrow H_i$ . It clear that  $(H_i, L_i)$  is a Hilbert complex and by its construction it follows immediately that  $(H_i, D_i)$  and  $(H_i, L_i)$  are a pair of related Hilbert complexes having the maps  $\{\phi_i\}$  as link maps. Finally it is clear that if each  $\phi_j$  is an isometry then the complexes  $(H_i, D_i)$  and  $(H_i, L_i)$  are complementary with  $\{\phi_i\}$  as link maps.  $\square$

Now we give the following definition which we will use later.

**Definition 3.** *Let  $V_0, V_1, \dots, V_n$  be a finite sequence of finite dimensional vector spaces. We will say that it is a finite sequence of finite dimensional vector spaces with Poincaré duality if for each  $i$ :*

$$V_i \cong V_{n-i}.$$

We are now in position to state the first of the two main results of this section.

**Theorem 7.** *Let  $(H_j, D_j) \subseteq (H_j, L_j)$  be a pair of complementary Hilbert complexes. Let  $i_{r,j}^*$  be the map defined in (15). Suppose that for each  $j$*

$$\text{im}(\overline{H}^j(H_*, D_*) \xrightarrow{i_{r,j}^*} \overline{H}^j(H_*, L_*)) \quad (16)$$

*is finite dimensional. Then*

$$\text{im}(\overline{H}^j(H_*, D_*) \xrightarrow{i_{r,j}^*} \overline{H}^j(H_*, L_*)), \quad j = 0, \dots, n \quad (17)$$

*is a finite sequence of finite dimensional vector spaces with Poincaré duality.*

Now we state some propositions which we will use in the proof of theorem 7.

**Proposition 7.** *Let  $H, K$  be two Hilbert spaces and let  $T : H \rightarrow K$  be a linear and continuous map. Let  $T^* : K \rightarrow H$  the adjoint of  $T$ . Suppose that  $\text{ran}(T)$  is closed. Then  $T : \text{Ker}(T)^\perp \rightarrow \text{Ker}(T^*)^\perp$  is continuous, bijective with bounded inverse.*

*Proof.* We have  $K = \text{Ker}(T^*) \oplus \text{Ker}(T^*)^\perp$  and  $\text{Ker}(T^*)^\perp = \overline{\text{ran}(T)}$ . Therefore by the fact that  $\text{ran}(T)$  is closed it follows that  $T$  is a bijection between  $\text{Ker}(T)^\perp$  and  $\text{Ker}(T^*)^\perp$ . Now from the fact that  $\text{Ker}(T)^\perp$  and  $(\text{Ker}(T^*))^\perp$  are closed subspace of  $H$  and  $K$  respectively it follows we can look at them as Hilbert spaces with the products induced by the products of  $H$  and  $K$  respectively. In this way we can use the closed graph theorem to conclude that  $T|_{\text{Ker}(T)^\perp}$  has a continuous inverse.  $\square$

**Proposition 8.** *Let  $H$  be a Hilbert space and let  $M, N$  be two closed subspace of it. Let  $\pi_M, \pi_N$  be the orthogonal projection on  $M$  and  $N$  respectively. Consider  $M$  and  $N$  as Hilbert spaces with the scalar product induced by the one of  $H$ . Then*

$$\pi_M|_N = (\pi_N|_M)^*$$

*that is if we look at  $\pi_N|_M$  as a linear and continuous map from the Hilbert space  $M$  to the Hilbert space  $N$  then  $\pi_M|_N$  is its adjoint.*

*Proof.* During the proof we use  $\langle, \rangle_H$  to indicate the scalar product of  $H$  and  $\langle, \rangle_M, \langle, \rangle_N$  to indicate the scalar product induced by  $\langle, \rangle_H$  on  $M$  and  $N$  respectively. For each  $u \in M$ ,  $v \in N$  we have  $\langle \pi_N(u), v \rangle_N = \langle \pi_N(u) + \pi_{N^\perp}(u), v \rangle_H = \langle u, v \rangle_H = \langle u, \pi_M(v) + \pi_{M^\perp}(v) \rangle_H = \langle u, \pi_M(v) \rangle_M$  and so we get the assertion.  $\square$

Now we are in position to prove theorem 7.

*Proof.* From proposition 1 we know that  $H_j = \mathcal{H}^j(H_*, D_*) \oplus \overline{\text{ran}(D_{j-1})} \oplus \overline{\text{ran}(D_j^*)}$  and that  $H_j = \mathcal{H}^j(H_*, L_*) \oplus \overline{\text{ran}(L_{j-1})} \oplus \overline{\text{ran}(L_j^*)}$ . So for each  $j$  we can define  $\pi_{D_j}$  as the orthogonal projection of  $H_j$  on  $\mathcal{H}^j(H_*, D_*)$  and  $\pi_{L_j}$  as the orthogonal projection of  $H_j$  on  $\mathcal{H}^j(H_*, L_*)$ . In the same way we can define  $\pi_{\overline{\text{ran}(D_{j-1})}}$ ,  $\pi_{\overline{\text{ran}(L_{j-1})}}$ ,  $\pi_{\overline{\text{ran}(D_j^*)}}$  and  $\pi_{\overline{\text{ran}(L_j^*)}}$ . Finally we define

$$\pi_{1,j} := (\pi_{L_j})|_{\mathcal{H}^j(H_*, D_*)}, \quad \pi_{2,j} := (\pi_{\overline{\text{ran}(L_{j-1})}})|_{\mathcal{H}^j(H_*, D_*)}, \quad \pi_{3,j} := (\pi_{\overline{\text{ran}(L_j^*)}})|_{\mathcal{H}^j(H_*, D_*)}.$$

Analogously, but now projecting from  $\mathcal{H}^j(H_*, L_*)$  on the orthogonal components of the sum  $H_j = \mathcal{H}^j(H_*, D_*) \oplus \overline{\text{ran}(D_{j-1})} \oplus \overline{\text{ran}(D_j^*)}$ , we define  $\pi_{4,j}, \pi_{5,j}, \pi_{6,j}$ . Our first claim is to show that for each  $j$

$$\pi_{1,j}(\mathcal{H}^j(H_*, D_*)) \cong \text{im}(\overline{H}^j(H_*, D_*) \xrightarrow{i_{r,j}^*} \overline{H}^j(H_*, L_*)) \quad (18)$$

Let  $[h] \in \overline{H}^j(H_*, D_*)$  a cohomology class. By (11) we know that there exists a unique representative of  $[h]$  in  $\mathcal{H}^j(H_*, D_*)$ . We call it  $\omega$ . Every other representative of  $[h]$  differs from  $\omega$  by an element in  $\overline{\text{ran}(D_{j-1})}$ ; therefore  $i_{r,j}^*([h]) = [i_j(\omega)]$ . Now we can decompose  $\omega$  as  $\omega = \pi_{1,j}(\omega) + \pi_{2,j}(\omega) + \pi_{3,j}(\omega)$ . Clearly  $[i_j(\omega)] = [\pi_{1,j}(\omega)] + [\pi_{3,j}(\omega)]$ . So if we show that  $\pi_{3,j}|_{\mathcal{H}^j(H_*, D_*)} \equiv 0$  we get the claim. Now let  $\eta \in \mathcal{H}^j(H_*, D_*)$ . Then  $\pi_{3,j}(\eta) \in \overline{\text{ran}(L_j^*)} \cap \text{Ker}(L_j)$  because  $\pi_{3,j}(\eta) = \eta - \pi_{1,j}(\eta) - \pi_{2,j}(\eta)$  and each term on the right hand side of the equality lies in  $\text{Ker}(L_j)$ . But  $(\text{Ker}(L_j))^\perp = \overline{\text{ran}(L_j^*)}$  and therefore  $\pi_{3,j}(\eta) = 0$ . So for each  $\eta \in \mathcal{H}^j(H_*, D_*)$  we have  $\pi_{3,j}(\eta) = 0$ . Therefore the claim is proved. In this way we know that  $\pi_{1,j}$  has closed range and that  $\text{Ker}(\pi_{1,j}) = \overline{\text{ran}(L_{j-1})} \cap \mathcal{H}^j(H_*, D_*)$ . In the same way it follows that  $\text{Ker}(\pi_{4,j}) = \overline{\text{ran}(D_j^*)} \cap \mathcal{H}^j(H_*, L_*)$ . Finally from the observations above and from propositions 7 and 8 we get:

1.  $\mathcal{H}^j(H_*, D_*) = \text{ran}(\pi_{4,j}) \oplus (\overline{\text{ran}(L_{j-1})} \cap \mathcal{H}^j(H_*, D_*)) = \text{ran}(\pi_{4,j}) \oplus \text{Ker}(\pi_{1,j})$  for each  $j$ .
2.  $\mathcal{H}^j(H_*, L_*) = \text{ran}(\pi_{1,j}) \oplus (\overline{\text{ran}(D_j^*)} \cap \mathcal{H}^j(H_*, L_*)) = \text{ran}(\pi_{1,j}) \oplus \text{Ker}(\pi_{4,j})$  for each  $j$ .
3.  $(\pi_{1,j})^* = \pi_{4,j}$  and both induce an isomorphism between  $\text{ran}(\pi_{4,j})$  and  $\text{ran}(\pi_{1,j})$ .

By the fourth point of proposition 5 it follows that each  $\phi_j$  induces an isomorphism between  $\mathcal{H}^j(H_*, D_*)$  and  $\mathcal{H}^{n-j}(H_*, L_*)$ . For the same reason  $\phi_j$  induces an isomorphism between  $\overline{\text{ran}(L_{j-1})}$  and  $\overline{\text{ran}(D_{n-j}^*)}$  and between  $\overline{\text{ran}(D_{j-1})}$  and  $\overline{\text{ran}(L_{n-j}^*)}$ . This implies that each  $\phi_j$  induces an isomorphism between  $\mathcal{H}^j(H_*, D_*) \cap \overline{\text{ran}(L_{j-1})}$  and  $\mathcal{H}^{n-j}(H_*, L_*) \cap \overline{\text{ran}(D_{n-j}^*)}$  that is an isomorphism between  $\text{Ker}(\pi_{1,j})$  and  $\text{Ker}(\pi_{4,n-j})$ . In this way we can conclude that each  $\phi_j$  induces an isomorphism between

$$\frac{\mathcal{H}^j(H_*, D_*)}{\text{Ker}(\pi_{1,j})} \quad \text{and} \quad \frac{\mathcal{H}^{n-j}(H_*, L_*)}{\text{Ker}(\pi_{4,n-j})}.$$

But

$$\frac{\mathcal{H}^j(H_*, D_*)}{\text{Ker}(\pi_{1,j})} \cong \text{ran}(\pi_{4,j}) \cong \text{ran}(\pi_{1,j}) \cong \text{im}(\overline{H}^j(H_*, D_*) \xrightarrow{i_{r,j}^*} \overline{H}^j(H_*, L_*))$$

and similarly

$$\frac{\mathcal{H}^{n-j}(H_*, L_*)}{\text{Ker}(\pi_{4,n-j})} \cong \text{ran}(\pi_{1,n-j}) \cong \text{im}(\overline{H}^{n-j}(H_*, D_*) \xrightarrow{i_{r,n-j}^*} \overline{H}^{n-j}(H_*, L_*)).$$

The composition of the above isomorphisms gives

$$\text{im}(\overline{H}^j(H_*, D_*) \xrightarrow{i_{r,j}^*} \overline{H}^j(H_*, L_*)) \cong \text{im}(\overline{H}^{n-j}(H_*, D_*) \xrightarrow{i_{r,n-j}^*} \overline{H}^{n-j}(H_*, L_*))$$

and this complete the proof.  $\square$

**Remark 1.** By the above proof it follows that given a pair of Hilbert complexes  $(H_*, D_*) \subseteq (H_*, L_*)$ , without any other assumption, the following isomorphism holds for each  $j$  :

$$\text{ran}(\pi_{1,j}) \cong \text{im}(\overline{H}^j(H_*, D_*) \xrightarrow{i_{r,j}^*} \overline{H}^j(H_*, L_*)). \quad (19)$$



Moreover when the sequences of vector spaces on the right hand side of the above equality isomorphism is finite dimensional we have

$$\mathcal{H}^j(H_*, D_*) \cap (\mathcal{H}^j(H_*, D_*) \cap \overline{\text{ran}(L_{j-1})})^\perp \cong (\mathcal{H}^j(H_*, L_*) \cap \overline{\text{ran}(D_j^*)})^\perp \cap \mathcal{H}^j(H_*, L_*)$$

that is

$$\text{ran}(\pi_{1,j}) \cong \text{ran}(\pi_{4,j}).$$

The following statements are immediate consequences of theorem 7.

**Corollary 3.** *Suppose that one of the two complexes of theorem 7 is Fredholm; then also the other complex is Fredholm and*

$$\text{im}(H^j(H_*, D_*) \longrightarrow H^j(H_*, L_*)), \quad j = 0, \dots, n \quad (20)$$

is a finite sequence of finite dimensional vector spaces with Poincaré duality. Moreover

$$\text{ran}(\pi_{1,j}) \cong \text{im}(H^j(H_*, D_*) \longrightarrow H^j(H_*, L_*)). \quad (21)$$

and

$$\mathcal{H}^j(H_*, D_*) \cap (\mathcal{H}^j(H_*, D_*) \cap \text{ran}(L_{j-1}))^\perp \cong (\mathcal{H}^j(H_*, L_*) \cap \text{ran}(D_j^*))^\perp \cap \mathcal{H}^j(H_*, L_*). \quad (22)$$

**Proposition 9.** *Let  $(H_*, D_*) \subseteq (H_*, L_*)$  a couple of complementary Hilbert complexes. Furthermore suppose that there is a third Hilbert complex  $(H_*, P_*)$  with the following properties:*

1.  $(H_*, D_*) \subseteq (H_*, P_*) \subseteq (H_*, L_*)$ .
2. *The reduced cohomology of  $(H_*, P_*)$  is finite dimensional.*

Then

$$\text{im}(\overline{H}^j(H_*, D_*) \xrightarrow{i_{r,j}^*} \overline{H}^j(H_*, L_*)), \quad j = 0, \dots, n$$

is a finite sequence of finite dimensional vector spaces with Poincaré duality.

*Proof.* The assertion is an immediate consequence of the following, simple fact. Let  $i_{1,j}$  be the natural inclusion of  $(H_*, D_*)$  in  $(H_*, P_*)$ , let  $i_{2,j}$  be the natural inclusion of  $(H_*, P_*)$  in  $(H_*, L_*)$  and finally let  $i_{3,j}$  be the natural inclusion of  $(H_*, D_*)$  in  $(H_*, L_*)$ . Obviously we have  $i_{3,j} = i_{2,j} \circ i_{1,j}$ . This implies that also the respective maps induced between the reduced cohomology groups commute. So we have  $i_{r,3,j}^* = i_{r,2,j}^* \circ i_{r,1,j}^*$  and therefore

$$\text{im}(\overline{H}^j(H_*, D_*) \xrightarrow{i_{r,3,j}^*} \overline{H}^j(H_*, L_*)) \subseteq \text{im}(\overline{H}^j(H_*, P_*) \xrightarrow{i_{r,2,j}^*} \overline{H}^j(H_*, L_*)).$$

In this way, by the second hypothesis, we know that

$$\text{im}(\overline{H}^j(H_*, D_*) \xrightarrow{i_{r,3,j}^*} \overline{H}^j(H_*, L_*))$$

is a finite sequence of finite dimensional vector spaces. Now we are in position to apply theorem 7 and so the proposition follows.  $\square$

Finally we conclude this sections with the following result:

**Theorem 8.** *Let  $(H_j, D_j) \subseteq (H_j, L_j)$   $j = 0, \dots, n$  a pair of Hilbert complexes. Suppose that for each  $j$   $\text{ran}(D_j)$  is closed in  $H_{j+1}$ . Then there exists a third Hilbert complex  $(H_j, P_j)$  such that*

1.  $(H_j, D_j) \subseteq (H_j, P_j) \subseteq (H_j, L_j)$ .
2.  $H^i(H_*, P_*) = \text{im}(H^i(H_*, D_*) \rightarrow H^i(H_*, L_*))$ .

Moreover if  $(H_j, D_j) \subseteq (H_j, L_j)$  are complementary and  $(H_j, D_j)$ , or equivalently  $(H_j, L_j)$ , is Fredholm then  $(H_j, P_j)$  is a Fredholm complex with Poincaré duality.

*Proof.* It is immediate that

$$\text{im}(H^i(H_*, D_*) \rightarrow H^i(H_*, L_*)) = \frac{\text{Ker}(D_i)}{\text{ran}(L_{i-1}) \cap \mathcal{D}(D_i)}.$$

Therefore for each  $i = 0, \dots, n$  we have to construct a closed extension of  $D_i$ , that we call  $P_i$ , such that  $\text{Ker}(P_i) = \text{Ker}(D_i)$  and  $\text{ran}(P_{i-1}) = \text{ran}(L_{i-1}) \cap \mathcal{D}(D_i)$ . To do this, from now on we will consider the following Hilbert space  $(\mathcal{D}(L_i), \langle \cdot, \cdot \rangle_{\mathcal{G}})$ , which is by definition the domain of  $L_i$  endowed with the graph scalar product. Therefore all the direct sum that will appear and all the assertions of topological type are referred to this Hilbert space  $(\mathcal{D}(L_i), \langle \cdot, \cdot \rangle_{\mathcal{G}})$ . We can decompose  $(\mathcal{D}(L_i), \langle \cdot, \cdot \rangle_{\mathcal{G}})$  in the following way:

$$(\mathcal{D}(L_i), \langle \cdot, \cdot \rangle_{\mathcal{G}}) = \text{Ker}(L_i) \oplus V_i \quad (23)$$

where  $V_i = \{\alpha \in \mathcal{D}(L_i) \cap \overline{\text{ran}(L_i^*)}\}$  and obviously these subspaces are both closed in  $(\mathcal{D}(L_i), \langle \cdot, \cdot \rangle_{\mathcal{G}})$ .

Consider now  $(\mathcal{D}(D_i), \langle \cdot, \cdot \rangle_{\mathcal{G}})$ ; it is a closed subspace of  $(\mathcal{D}(L_i), \langle \cdot, \cdot \rangle_{\mathcal{G}})$  and we can decompose it as

$$(\mathcal{D}(D_i), \langle \cdot, \cdot \rangle_{\mathcal{G}}) = \text{Ker}(D_i) \oplus A_i. \quad (24)$$

Analogously to the previous case  $A_i = \{\alpha \in \mathcal{D}(D_i) \cap \overline{\text{ran}(D_i^*)}\}$  and obviously these subspaces are both closed in  $(\mathcal{D}(D_i), \langle \cdot, \cdot \rangle_{\mathcal{G}})$ . Now let  $C_i = \{\alpha \in \mathcal{D}(L_i) : L_i(\alpha) \in \mathcal{D}(D_{i+1})\}$ .  $C_i$  is closed in  $(\mathcal{D}(D_i), \langle \cdot, \cdot \rangle_{\mathcal{G}})$  because it is the preimage of a closed subspace under a continuous map. Finally let  $W_i = C_i \cap V_i$ . Then it is clear that

$$C_i = \text{Ker}(L_i) \oplus W_i. \quad (25)$$

Obviously if  $\text{Ker}(D_i) = \text{Ker}(L_i)$  then it enough to define  $P_i := L_i|_{C_i}$ . So we can suppose that  $\text{Ker}(D_i)$  is properly contained in  $\text{Ker}(L_i)$ . Let  $\pi_1$  be the orthogonal projection of  $A_i$  onto  $\text{Ker}(L_i)$  and analogously let  $\pi_2$  be the orthogonal projection of  $A_i$  onto  $V_i$ . We have the following properties:

1.  $\pi_2$  is injective
2.  $\text{ran}(\pi_2) \subseteq W_i$
3.  $\text{ran}(\pi_2)$  is closed.

The first property follows from the fact that  $\text{Ker}(\pi_2) = A_i \cap \text{Ker}(L_i)$ . But  $L_i$  is an extension of  $D_i$ ; therefore if an element  $\alpha$  lies in  $A_i \cap \text{Ker}(L_i)$  then it lies also in  $\text{Ker}(D_i)$  and so  $\alpha = 0$ . For the second property, given  $\alpha \in A_i$ , we have  $D_i(\alpha) = L_i(\alpha) = L_i(\pi_1(\alpha) + \pi_2(\alpha)) = L_i(\pi_2(\alpha))$  and therefore  $\pi_2(\alpha) \in W_i$ . Finally, for the third property, consider a sequence  $\{\gamma_m\}_{m \in \mathbb{N}} \subset A_i$  such that  $\pi_2(a_m)$  converges to  $\gamma \in W_i$ . Then

$$\lim_{m \rightarrow \infty} D_j(a_m) = \lim_{m \rightarrow \infty} L_j(a_m) = \lim_{m \rightarrow \infty} L_j(\pi_2(a_m)) = L_j(\gamma).$$

This implies that

$$\lim_{m \rightarrow \infty} D_j(a_m) = L_j(\gamma)$$

and therefore the limit exists. So by the assumptions about the range of  $D_j$  it follows that there exists an element  $\eta \in A_i$  such that

$$\lim_{m \rightarrow \infty} D_j(a_m) = D_j(\eta).$$

Moreover  $L_j(\gamma) = D_j(\eta) = L_j(\eta) = L_j(\pi_2(\eta))$ . This implies that  $L_j(\pi_2(\eta) - \gamma) = 0$  and therefore  $\pi_2(\eta) = \gamma$  because  $\pi_2(\eta), \gamma \in W_i$  and  $L_i$  is injective on  $W_i$ . In this way we showed that  $\pi_2$  is closed.

Now define  $N_i$  as the orthogonal complement of  $\text{ran}(\pi_2)$  in  $W_i$ . Then for each  $\alpha \in A_i$  and for each  $\beta \in N_i$  we have  $\langle \alpha, \beta \rangle_{\mathcal{G}} = \langle \pi_1(\alpha) + \pi_2(\alpha), \beta \rangle_{\mathcal{G}} = 0$ . This last property, joined with the fact that both  $A_i$  and  $N_i$  are closed, implies that the vector space generated by  $A_i$  and  $N_i$  is closed and, if we call it  $M_i$ , then we have  $M_i = A_i \oplus N_i$ . Again for each  $\alpha \in \text{Ker}(D_i)$  and for each  $\beta \in M_i$  we have  $\langle \alpha, \beta \rangle_{\mathcal{G}} = 0$ . This is because for each  $\beta \in M_i$  there exist unique

$\beta_1 \in A_i$ ,  $\beta_2 \in N_i$  such that  $\beta = \beta_1 \oplus \beta_2$ . Now it is clear that  $\langle \alpha, \beta_1 \rangle_{\mathcal{G}} = 0 = \langle \alpha, \beta_2 \rangle_{\mathcal{G}}$  because  $\text{Ker}(D_i) \subset \text{Ker}(L_i)$ ,  $N_i \subset W_i$ ,  $W_i$  and  $\text{Ker}(L_i)$  are orthogonal and  $\text{Ker}(D_i)$  and  $A_i$  are orthogonal. Therefore, also in this case, if we call  $B_i$  the vector space generated by  $\text{Ker}(D_i)$  and  $M_i$  we have that  $B_i = \text{Ker}(D_i) \oplus M_i = \mathcal{D}(D_i) \oplus N_i$  and therefore  $B_i$  is closed. Finally define  $P_i$  as

$$P_i := L_i|_{B_i} \quad (26)$$

By the construction it is clear that for each  $\alpha \in B_i$  then  $P_i(\alpha) \in \mathcal{D}(D_{i+1}) \cap \text{ran}(L_i)$  and that  $\mathcal{D}(D_i) \subset B_i$ . Therefore this implies that the composition  $P_{i+1} \circ P_i$  is defined on the whole  $B_i$  and that  $P_{i+1} \circ P_i \equiv 0$ . Moreover, if we look at  $P_i$  as an unbounded operator from  $H_i$  to  $H_{i+1}$ , then it is clear that it is densely defined and closed.

To conclude the proof we have to check that  $\text{Ker}(P_i) = \text{Ker}(D_i)$  and that  $\text{ran}(P_i) = \text{ran}(L_i) \cap \mathcal{D}(D_{i+1})$ . Let  $\alpha \in \text{Ker}(P_i)$ . We can decompose  $\alpha$  in a unique way as  $\alpha_1 + \alpha_2 + \alpha_3$  where  $\alpha_1 \in \text{Ker}(D_i)$ ,  $\alpha_2 \in A_i$  and  $\alpha_3 \in N_i$ . The assumption on  $\alpha$  implies that  $\alpha_2 + \alpha_3 \in \text{Ker}(P_i)$ . We can decompose  $\alpha_2$  in a unique way as  $\alpha_2 = \beta_1 + \beta_2$  where  $\beta_1 \in \text{ran}(\pi_1)$  and  $\beta_2 \in \text{ran}(\pi_2)$ . Again from the assumption on  $\alpha$  it follows that  $L_i(\beta_2 + \alpha_3) = 0$ . This implies that  $\beta_2 + \alpha_3 \in W_i \cap \text{Ker}(L_i)$  and therefore from (25) we can conclude that  $\beta_2 + \alpha_3 = 0$ . But  $\beta_2 + \alpha_3 \in \text{ran}(\pi_2) \oplus N_i$ ,  $\beta_2 \in \text{ran}(\pi_2)$ ,  $\alpha_3 \in N_i$  and so we got  $0 = \beta_2 = \alpha_3$ . Now we have  $\alpha_2 = \beta_1$  that is  $\alpha_2 \in A_i \cap \text{Ker}(L_i) = \text{Ker}(\pi_2)$ . By the injectivity of  $\pi_2$  it follows that  $\alpha_2 = 0$  and therefore  $\alpha = \alpha_1 \in \text{Ker}(D_i)$ . So we got  $\text{Ker}(P_i) \subseteq \text{Ker}(D_i)$ ; the other inclusion is trivial and therefore we have  $\text{Ker}(P_i) = \text{Ker}(D_i)$ . Now we have to check that  $\text{ran}(P_i) = \text{ran}(L_i) \cap \mathcal{D}(D_{i+1})$ . Clearly, as observed above, the inclusion  $\subseteq$  follows immediately by the construction of  $P_i$ . So we have to prove the converse. Let  $\gamma \in W_i$ . Then there exist and are unique  $\gamma_1 \in \text{ran}(\pi_2)$  and  $\gamma_2 \in N_i$  such that  $\gamma = \gamma_1 + \gamma_2$ . Now let  $\theta \in A_i$  be the unique element in  $A_i$  such that  $\pi_2(\theta) = \gamma_1$ . Finally consider  $\theta + \gamma_2$ . Then  $\theta + \gamma_2 \in B_i$  and  $P_i(\theta + \gamma_2) = L_i(\theta + \gamma_2) = L_i(\pi_1(\theta) + \pi_2(\theta) + \gamma_2) = L_i(\gamma_1 + \gamma_2) = L_i(\gamma)$ .

In this way we showed that  $\text{ran}(L_i) \cap \mathcal{D}(D_{i+1}) = \text{ran}(P_i)$ .

Finally if  $(H_j, D_j)$  or equivalently  $(H_j, L_j)$  is Fredholm then  $H^i(H_*, D_*)$  is finite dimensional for each  $i$  and therefore  $\text{ran}(D_j)$  is closed in  $H_{j+1}$  for each  $j$ . We have the following natural and surjective map

$$\frac{\text{Ker}(D_{i+1})}{\text{ran}(D_i)} \longrightarrow \frac{\text{Ker}(D_{i+1})}{\text{ran}(P_i)}.$$

This implies that also  $H^i(H_*, P_*)$  is finite dimensional, that is  $(H_j, P_j)$  is a Fredholm complex, and now using theorem 7 it follows that Poincaré duality holds for it. This complete the proof.  $\square$

## 2 Geometric Applications

### 2.1 Duality for reduced $L^2$ cohomology

Now we want to show that riemannian geometry is a context in which pairs of complementary Hilbert complexes appear in a natural way.

Let  $(M, g)$  be an open and oriented riemannian manifold of dimension  $m$  and let  $E_0, \dots, E_n$  be vector bundles over  $M$ . For each  $i = 0, \dots, n$  let  $C_c^\infty(M, E_i)$  be the space of smooth section with compact support. If we put on each vector bundle a metric  $h_i$   $i = 0, \dots, n$  then we can construct in a natural way a sequences of Hilbert space  $L^2(M, E_i)$ ,  $i = 0, \dots, n$  as the completion of  $C_c^\infty(M, E_i)$ . Now suppose that we have a complex of differential operators :

$$0 \rightarrow C_c^\infty(M, E_0) \xrightarrow{P_0} C_c^\infty(M, E_1) \xrightarrow{P_1} C_c^\infty(M, E_2) \xrightarrow{P_2} \dots \xrightarrow{P_{n-1}} C_c^\infty(M, E_n) \rightarrow 0, \quad (27)$$

To turn this complex into a Hilbert complex we must specify a closed extension of  $P_*$  that is an operator between  $L^2(M, E_*)$  and  $L^2(M, E_{*+1})$  with closed graph which is an extension of  $P_*$ . So we state some definitions and propositions which generalize those ones stated in [4]. We start recalling the two canonical closed extensions of  $P$ .

**Definition 4.** *The maximal extension  $P_{\max}$ ; this is the operator acting on the domain:*

$$\mathcal{D}(P_{\max, i}) = \{\omega \in L^2(M, E_i) : \exists \eta \in L^2(M, E_{i+1}) \quad (28)$$

$$s.t. \quad \langle \omega, P_i^t \zeta \rangle_{L^2(M, E_i)} = \langle \eta, \zeta \rangle_{L^2(M, E_{i+1})} \quad \forall \zeta \in C_0^\infty(M, E_{i+1})\}$$

where  $P_i^t$  is the formal adjoint of  $P_i$ .

In this case  $P_{max,i}\omega = \eta$ . In other words  $\mathcal{D}(P_{max,i})$  is the largest set of forms  $\omega \in L^2(M, E_i)$  such that  $P_i\omega$ , computed distributionally, is also in  $L^2(M, E_{i+1})$ .

**Definition 5.** The minimal extension  $P_{min,i}$ ; this is given by the graph closure of  $P_i$  on  $C_0^\infty(M, E_i)$  respect to the norm of  $L^2(M, E_i)$ , that is,

$$\mathcal{D}(P_{min,i}) = \{\omega \in L^2(M, E_i) : \exists \{\omega_j\}_{j \in J} \subset C_0^\infty(M, E_i), \omega_j \rightarrow \omega, P_i\omega_j \rightarrow \eta \in L^2(M, E_{i+1})\} \quad (29)$$

and in this case  $P_{min,i}\omega = \eta$

Obviously  $\mathcal{D}(P_{min,i}) \subset \mathcal{D}(P_{max,i})$ . Furthermore, from these definitions, it follows immediately that

$$P_{min,i}(\mathcal{D}(P_{min,i})) \subset \mathcal{D}(P_{min,i+1}), \quad P_{min,i+1} \circ P_{min,i} = 0$$

and that

$$P_{max,i}(\mathcal{D}(P_{max,i})) \subset \mathcal{D}(P_{max,i+1}), \quad P_{max,i+1} \circ P_{max,i} = 0.$$

Therefore  $(L^2(M, E_*), P_{max/min,*})$  are both Hilbert complexes and their cohomology groups, reduced cohomology groups, are denoted respectively by  $H_{2,max/min}^i(M, E_*)$  and  $\overline{H}_{2,max/min}^i(M, E_*)$ .

Another straightforward but important fact is that the Hilbert complex adjoint of  $(L^2(M, E_*), P_{max/min,*})$  is  $(L^2(M, E_*), P_{min/max,*}^t)$ , that is

$$(P_{max,i})^* = P_{min,i}^t, \quad (P_{min,i})^* = P_{max,i}^t. \quad (30)$$

Using proposition 1 we obtain two weak Kodaira decompositions:

$$L^2(M, E_i) = \mathcal{H}_{abs/rel}^i(M, E_i) \oplus \overline{\text{ran}(P_{max/min,i-1})} \oplus \overline{\text{ran}(P_{min/max,i}^t)} \quad (31)$$

with summands mutually orthogonal in each case. For the first summand in the right, called the absolute or relative Hodge cohomology, we have by (10):

$$\mathcal{H}_{abs/rel}^i(M, E_*) = \text{Ker}(P_{max/min,i}) \cap \text{Ker}(P_{min/max,i-1}^t). \quad (32)$$

We can also consider the two natural Laplacians associated to these Hilbert complexes, that is for each  $i$

$$P_{min,i}^t \circ P_{max,i} + P_{max,i-1} \circ P_{min,i-1}^t \quad (33)$$

and

$$P_{max,i}^t \circ P_{min,i} + P_{min,i-1} \circ P_{max,i-1}^t \quad (34)$$

with domain described in (9). Using (10) and (11) it follows that the nullspace of (33) is isomorphic to the absolute Hodge cohomology which is in turn isomorphic to the reduced cohomology of the Hilbert complex  $(L^2(M, E_*), P_{max,*})$ . Analogously, using again (10) and (11), it follows that the nullspace of (34) is isomorphic to the relative Hodge cohomology which is in turn isomorphic to the reduced cohomology of the Hilbert complex  $(L^2(M, E_*), P_{min,*})$ . Finally we recall that we can define other two Hodge cohomology groups  $\mathcal{H}_{max,min}^i(M, E_*)$  defined as

$$\mathcal{H}_{max,min}^i(M, E_*) = \text{Ker}(P_{max/min,i}) \cap \text{Ker}(P_{max/min,i-1}^t). \quad (35)$$

Now we are in position to state the following results:

**Theorem 9.** Let  $(M, g)$  be an open and oriented riemannian manifold of dimension  $m$  and let  $E_0, \dots, E_n$  be vector bundles over  $M$  endowed with metrics  $h_i$   $i = 0, \dots, n$ . Suppose that we have a complex of differential operator :

$$0 \rightarrow C_c^\infty(M, E_0) \xrightarrow{P_0} C_c^\infty(M, E_1) \xrightarrow{P_1} C_c^\infty(M, E_2) \xrightarrow{P_2} \dots \xrightarrow{P_{n-1}} C_c^\infty(M, E_n) \rightarrow 0, \quad (36)$$

and let

$$0 \rightarrow L^2(M, E_0) \xrightarrow{P_{max,0}} L^2(M, E_1) \xrightarrow{P_{max,1}} L^2(M, E_2) \xrightarrow{P_{max,2}} \dots \xrightarrow{P_{max,n-1}} L^2(M, E_n) \rightarrow 0, \quad (37)$$

and

$$0 \rightarrow L^2(M, E_0) \xrightarrow{P_{min,0}^t} L^2(M, E_1) \xrightarrow{P_{min,1}^t} L^2(M, E_2) \xrightarrow{P_{min,2}^t} \dots \xrightarrow{P_{min,n-1}^t} L^2(M, E_n) \rightarrow 0, \quad (38)$$

the two natural Hilbert complexes associated to (36) as described above. Suppose that for each  $i = 0, \dots, n$  there exists an isometry  $\phi_i : (E_i, h_i) \rightarrow (E_{n-i}, h_{n-i})$ ; with a little abuse of notation let still  $\phi_i$  denotes the induced isometry from  $L^2(M, E_i)$  to  $L^2(M, E_{n-i})$ . Finally suppose that  $P_{n-i-1}^t \circ \phi_i = c_i(\phi_{i+1} \circ P_i)$ , where  $c_i$  is a constant which depends only on  $i$ .

If  $\text{im}(\overline{H}_{2,min}^i(M, E_*) \xrightarrow{i_{r,i}^*} \overline{H}_{2,max}^i(M, E_*))$  is finite dimensional for each  $i$  then

$$\text{im}(\overline{H}_{2,min}^i(M, E_*) \xrightarrow{i_{r,i}^*} \overline{H}_{2,max}^i(M, E_*))$$

is a finite sequence of finite dimensional vector spaces with Poincaré duality.

*Proof.* From the hypothesis we know that for each  $i = 0, \dots, n$  there exists an isometry  $\phi_i : (E_i, h_i) \rightarrow (E_{n-i}, h_{n-i})$  such that  $P_{n-i-1}^t \circ \phi_i = c_i(\phi_{i+1} \circ P_i)$ , where  $c_i$  is a constant which depends just by  $i$ . This isometries of vector bundles induces isometries from  $L^2(M, E_i)$  to  $L^2(M, E_{n-i})$ , that with a little abuse of notation we still label  $\phi_i$ , such that  $\phi_i(\mathcal{D}(P_{min,i})) = \mathcal{D}(P_{min,n-i-1}^t)$  and  $P_{min,n-i-1}^t \circ \phi_i = c_i(\phi_{i+1} \circ P_{min,i})$ . So we showed that the complexes  $(L^2(M, E_*), P_{min,*}) \subseteq (L^2(M, E_*), P_{max,*})$  are a pair of complementary Hilbert complexes. Now, applying theorem 7, we can get the conclusion.  $\square$

**Theorem 10.** *In the same hypothesis of the previous theorem, suppose furthermore that for each  $i = 0, \dots, n$   $\text{ran}(P_{min,i})$  is closed in  $L^2(M, E_{i+1})$ . Then there exists a Hilbert complex  $(L^2(M, E_i), P_{m,i})$  such that for each  $i = 0, \dots, n$*

$$\mathcal{D}(P_{min,i}) \subset \mathcal{D}(P_{m,i}) \subset \mathcal{D}(P_{max,i}),$$

$P_{max,i}$  is an extension of  $P_{m,i}$  which is an extension of  $P_{min,i}$  and

$$H_{2,m}^i(M, E_i) = \text{im}(H_{2,min}^i(M, E_*) \xrightarrow{i_i^*} H_{2,max}^i(M, E_*))$$

where  $H_{2,m}^i(M, E_i)$  is the cohomology of the Hilbert complex  $(L^2(M, E_i), P_{m,i})$ . Finally if  $(L^2(M, E_i), P_{max,i})$  or equivalently  $(L^2(M, E_i), P_{min,i})$  is Fredholm then  $(L^2(M, E_i), P_{m,i})$  is a Fredholm complex with Poincaré duality.

*Proof.* It follows immediately from the previous theorem and from theorem 8.  $\square$

As a particular and important case we have the following two theorems:

**Theorem 11.** *Let  $(M, g)$  an open, oriented and incomplete riemannian manifold of dimension  $m$ . Then the complexes*

$$(L^2\Omega^*(M, g), d_{max,*}) \text{ and } (L^2\Omega^*(M, g), d_{min,*})$$

are a pair of complementary Hilbert complexes.

In particular if  $\text{im}(\overline{H}_{2,min}^i(M, g) \xrightarrow{i_{r,i}^*} \overline{H}_{2,max}^i(M, g))$  is finite dimensional for each  $i$  then

$$\text{im}(\overline{H}_{2,min}^i(M, g) \xrightarrow{i_{r,i}^*} \overline{H}_{2,max}^i(M, g))$$

is a finite sequence of finite dimensional vector spaces with Poincaré duality.

*Proof.* Let  $*$  :  $\Lambda^i(M) \rightarrow \Lambda^{n-i}(M)$  the Hodge star operator. Then  $*$  induces a map between  $\Omega_c^i(M)$  and  $\Omega_c^{n-i}(M)$  such that for  $\eta, \omega \in \Omega_c^i(M)$  we have:

$$\begin{aligned} \langle *\eta, *\omega \rangle_{L^2(M,g)} &= \int_M \langle *\eta, *\omega \rangle_M d\text{vol}_M = \int_M *\eta \wedge **\omega = \int_M \omega \wedge *\eta = \\ &= \langle \omega, \eta \rangle_{L^2(M,g)} = \langle \eta, \omega \rangle_{L^2(M,g)} \end{aligned}$$

that is  $*$  is an isometry between  $\Omega_c^i(M)$  and  $\Omega_c^{n-i}(M)$ . This implies that  $*$  extends to an isometry between  $L^2\Omega^i(M, g)$  and  $L^2\Omega^{n-i}(M, g)$ . Now it is an immediate consequence of definition 4 and definition 5 that

$$*d_{min,i} = \pm \delta_{min,n-i-1} * \quad \text{and that} \quad *d_{max,i} = \pm \delta_{max,n-i-1} *$$

and the sign depends only on the parity of the degree  $i$ . So we can apply theorem 7 and the assertion follows.  $\square$

**Remark 2.** *The previous theorem shows that pair of complementary Hilbert complexes appear naturally in riemannian geometry. In fact the Hodge star operator provides naturally a family of link maps and so we do not need to assume their existence.*

**Theorem 12.** *Let  $(M, g)$  be an open, oriented and incomplete riemannian manifold of dimension  $n$ . Suppose that for each  $i = 0, \dots, n$   $\text{ran}(d_{min,i})$  is closed in  $L^2\Omega^{i+1}(M, g)$ . Then there exists a Hilbert complex  $(L^2\Omega^i(M, g), d_{m,i})$  such that for each  $i = 0, \dots, n$*

$$\mathcal{D}(d_{min,i}) \subset \mathcal{D}(d_{m,i}) \subset \mathcal{D}(d_{max,i}),$$

$d_{max,i}$  is an extension of  $d_{m,i}$  which is an extension of  $d_{min,i}$  and

$$H_{2,m}^i(M, g) = \text{im}(H_{2,min}^i(M, g) \xrightarrow{i_i^*} H_{2,max}^i(M, g))$$

where  $H_{2,m}^i(M, g)$  is the cohomology of the Hilbert complex  $(L^2\Omega^i(M, g), d_{m,i})$ . Finally, if  $(L^2\Omega^i(M, g), d_{max,i})$  or equivalently  $(L^2\Omega^i(M, g), d_{min,i})$  is Fredholm, then  $(L^2\Omega^i(M, g), d_{m,i})$  is a Fredholm complex with Poincaré duality.

*Proof.* Also in this case it follows immediately from the previous theorem and from theorem 8.  $\square$

We have the following corollary:

**Corollary 4.** *In the same assumptions of theorem 12; Let  $\Delta_i : \Omega_c^i(M) \rightarrow \Omega_c^i(M)$  be the Laplacian acting on the space of smooth compactly supported forms. Then there exists a self-adjoint extension  $\Delta_{m,i} : L^2\Omega^i(M, g) \rightarrow L^2\Omega^i(M, g)$  with closed range such that*

$$\text{Ker}(\Delta_{m,i}) \cong \text{im}(H_{2,min}^i(M, g) \xrightarrow{i_i^*} H_{2,max}^i(M, g)).$$

Moreover, if  $(L^2\Omega^i(M, g), d_{max,i})$  or equivalently  $(L^2\Omega^i(M, g), d_{min,i})$  is Fredholm, then  $\Delta_{m,i}$  is a Fredholm operator on its domain endowed with the graph norm.

*Proof.* Consider the Hilbert complex  $(L^2\Omega^i(M, g), d_{m,i})$ . For each  $i = 0, \dots, n$  define

$$\Delta_{m,i} := d_{m,i}^* \circ d_{m,i} + d_{m,i-1} \circ d_{m,i-1}^* \quad (39)$$

with domain given by

$$\mathcal{D}(\Delta_{m,i}) = \{\omega \in \mathcal{D}(d_{m,i}) \cap \mathcal{D}(d_{m,i-1}^*) : d_{m,i}(\omega) \in \mathcal{D}(d_{m,i}^*) \text{ and } d_{m,i-1}^*(\omega) \in \mathcal{D}(d_{m,i-1})\}. \quad (40)$$

In other words, for each  $i = 0, \dots, n$ ,  $\Delta_{m,i}$  is the  $i$ -th Laplacian associated to the Hilbert complex  $(L^2\Omega^i(M, g), d_{m,i})$ . So, as recalled in the first section, it follows that (39) is a self-adjoint operator. Moreover, by the fact that  $d_{min,i}$  has closed range for each  $i = 0, \dots, n$  it follows that also  $\delta_{min,i}$  has closed range for each  $i$ . Finally this implies that also  $d_{max,i}$  has closed range because  $d_{max,i} = \delta_{min,i}^*$ . This means that for the Hilbert complex  $(L^2\Omega^i(M, g), d_{m,i})$  the  $L^2$  cohomology and the reduced  $L^2$  cohomology are exactly the same. The reason is that  $\overline{\text{ran}(d_{m,i})} = \overline{\text{ran}(d_{max,i}) \cap \text{Ker}(d_{min,i+1})} = \overline{\text{ran}(d_{max,i}) \cap \text{Ker}(d_{min,i+1})}$  because they are both closed in  $L^2\Omega^{i+1}(M, g)$  and clearly  $\text{ran}(d_{max,i}) \cap \text{Ker}(d_{min,i+1}) = \text{ran}(d_{m,i})$ . So we can apply (11) to get the first conclusion. Moreover by the fact that  $\text{ran}(\Delta_{m,i}) = \text{ran}(d_{m,i-1}) \oplus \text{ran}(\delta_{m,i})$  it follows that  $\Delta_{m,i}$  is an operator with closed range. Finally, using the fact that  $(L^2\Omega^i(M, g), d_{m,i})$  is Fredholm, we get that  $\Delta_{m,i}$  is self-adjoint, with finite dimensional nullspace and with closed range and therefore it is a Fredholm operator on its domain endowed with the graph norm.  $\square$

**Remark 3.** We remark that from the previous proof it follows that, under the assumptions of theorem 12, the operator  $d_{\mathfrak{m},i}$  has closed range for each  $i$  and therefore for the Hilbert complex  $(L^2\Omega^i(M, g), d_{\mathfrak{m},i})$  the  $L^2$  cohomology coincides with the reduced  $L^2$  cohomology.

From now on we will focus our attention exclusively on the vector spaces

$\text{im}(\overline{H}_{2,\min}^i(M, g) \xrightarrow{i_{r,i}^*} \overline{H}_{2,\max}^i(M, g))$  because, using these, we will get some geometric and topological applications concerning the manifold  $M$ .

Anyway it will be clear that all the following corollaries of the remaining part of this subsection apply also for the vector spaces  $\text{im}(\overline{H}_{2,\min}^i(M, E_*) \xrightarrow{i_{r,i}^*} \overline{H}_{2,\max}^i(M, E_*))$  under the hypothesis of theorem 9.

Now, to get a lighter notation, we label the vector spaces

$$\text{im}(\overline{H}_{2,\min}^i(M, g) \xrightarrow{i_{r,i}^*} \overline{H}_{2,\max}^i(M, g)) := \overline{H}_{2,m \rightarrow M}^i(M, g) \text{ and } H_{2,m \rightarrow M}^i(M, g)$$

in the non reduced case. Moreover, when it makes sense, we define

$$\overline{\chi}_{2,m \rightarrow M}(M, g) := \sum_{i=0}^m (-1)^i \dim(\overline{H}_{2,m \rightarrow M}^i(M, g)) \quad (41)$$

and in the non reduced case :

$$\chi_{2,m \rightarrow M}(M, g) := \sum_{i=0}^m (-1)^i \dim(H_{2,m \rightarrow M}^i(M, g)) \quad (42)$$

We have the following propositions:

**Proposition 10.** In the hypothesis of theorem 11, if  $m$  is odd then

$$\overline{\chi}_{2,m \rightarrow M}(M, g) = 0. \quad (43)$$

If  $m$  is even then

$$\overline{\chi}_{2,m \rightarrow M}(M, g) = \dim(\overline{H}_{2,m \rightarrow M}^{\frac{m}{2}}(M, g)) \quad (44)$$

when  $\frac{m}{2}$  is still even while if  $\frac{m}{2}$  is odd then

$$\overline{\chi}_{2,m \rightarrow M}(M, g) = -\dim(\overline{H}_{2,m \rightarrow M}^{\frac{m}{2}}(M, g)). \quad (45)$$

Finally if  $(L^2\Omega^i(M, g), d_{\max,i})$  is Fredholm, or equivalently if  $(L^2\Omega^i(M, g), d_{\min,i})$  is Fredholm, the above results holds for  $\chi_{2,m \rightarrow M}(M, g)$  and  $H_{2,m \rightarrow M}^{\frac{m}{2}}(M, g)$ .

*Proof.* The equalities (43), (44) and (45) are an immediate consequence of theorem 11. Finally, if for example  $(L^2\Omega^i(M, g), d_{\max,i})$  is Fredholm then  $H_{2,\max}^i(M, g) \cong \overline{H}_{2,\max}^i(M, g) \cong \overline{H}_{2,\min}^{n-i} \cong H_{2,\min}^{n-i}(M, g)$  and so also  $(L^2\Omega^i(M, g), d_{\min,i})$  is Fredholm. Obviously the same arguments show that, if  $(L^2\Omega^i(M, g), d_{\min,i})$  is Fredholm, then also  $(L^2\Omega^i(M, g), d_{\max,i})$  is Fredholm and therefore in (43), (44) and (45) we can use  $\chi_{2,M \rightarrow M}(M, g)$  and  $H_{2,m \rightarrow M}^{\frac{m}{2}}(M, g)$ .  $\square$

**Proposition 11.** In the hypothesis of theorem 7. Suppose that one of the two following properties is satisfied

1.  $i_{r,i}^* : \overline{H}_{2,\min}^i(M, g) \longrightarrow \overline{H}_{2,\max}^i(M, g)$  is injective,
2.  $i_{r,i}^* : \overline{H}_{2,\min}^i(M, g) \longrightarrow \overline{H}_{2,\max}^i(M, g)$  is surjective.

Then

$$\overline{H}_{2,\min}^i(M, g), \overline{H}_{2,\max}^i(M, g) \quad i = 0, \dots, n \quad (46)$$

are both a finite sequences of finite dimensional vector spaces with Poincaré duality. Finally, under the same hypothesis, if one of the two complexes  $(L^2\Omega^i(M, g), d_{\max/\min,i})$  is Fredholm then the same conclusion holds for

$$H_{2,\min}^i(M, g), H_{2,\max}^i(M, g) \quad i = 0, \dots, n$$

*Proof.* If  $i_{r,i}^* : \overline{H}_{2,min}^i(M, g) \longrightarrow \overline{H}_{2,max}^i(M, g)$  is injective then  $\overline{H}_{2,min}^i(M, g) \cong \overline{H}_{2,m \rightarrow M}^i(M, g)$ . This implies that each  $\overline{H}_{2,min}^i(M, g)$  is finite dimensional and therefore, using theorem 11, we get  $\overline{H}_{2,min}^i(M, g) \cong \overline{H}_{2,min}^{n-i}(M, g)$ . Finally by the fact that the Hodge star operator induces an isomorphism between  $\overline{H}_{2,min}^i(M, g)$  and  $\overline{H}_{2,max}^{n-i}(M, g)$  it follows that  $\overline{H}_{2,max}^i(M, g)$  is a finite sequences of finite dimensional vector spaces with Poincaré duality. In the same way if  $i_{r,i}^* : \overline{H}_{2,min}^i(M, g) \longrightarrow \overline{H}_{2,max}^i(M, g)$  is surjective then  $\overline{H}_{2,max}^i(M, g) \cong \overline{H}_{2,m \rightarrow M}^i(M, g)$ . Now the same arguments used in the injective case shows that  $\overline{H}_{2,max}^i(M, g)$  is a finite sequence of finite dimensional spaces with Poincaré duality. Finally, using again the isomorphism induced by the Hodge star operator between  $\overline{H}_{2,min}^i(M, g)$  and  $\overline{H}_{2,max}^{n-i}(M, g)$  we get the same conclusions for  $\overline{H}_{2,min}^i(M, g)$ .  $\square$

Finally we conclude the section with the following proposition; before stating it we give some definitions: let

$$d_m + d_m^* : \bigoplus_{i=0}^n L^2 \Omega^i(M, g) \longrightarrow \bigoplus_{i=0}^n L^2 \Omega^i(M, g) \quad (47)$$

the operator defined as  $d_m + d_m^*|_{L^2 \Omega^i(M, g)} = d_{m,i} + d_{m,i-1}^*$  where  $d_{m,i}$  is defined in theorem 12 and the domain of (47) is

$$\mathcal{D}(d_m + d_m^*) = \bigoplus_{i=0}^n \mathcal{D}(d_{m,i} + d_{m,i-1}^*)$$

and  $\mathcal{D}(d_{m,i} + d_{m,i-1}^*) = \mathcal{D}(d_{m,i}) \cap \mathcal{D}(d_{m,i-1}^*)$ .

**Proposition 12.** *Let  $(M, g)$  be an open oriented and incomplete riemannian manifold of dimension  $n$ . Suppose that for each  $i = 0, \dots, n$   $\text{ran}(d_{min,i})$  is closed in  $L^2 \Omega^{i+1}(M, g)$  and that  $(L^2 \Omega^i(M, g), d_{m,i})$  is a Fredholm complex. Then the operator  $(d_m + d_m^*)_{ev}$  defined as*

$$d_m + d_m^* : \bigoplus_{i=0}^n L^2 \Omega^{2i}(M, g) \longrightarrow \bigoplus_{i=0}^n L^2 \Omega^{2i+1}(M, g)$$

with domain given by

$$\mathcal{D}((d_m + d_m^*)_{ev}) := \mathcal{D}(d_m + d_m^*) \cap \bigoplus_{i=0}^n L^2 \Omega^{2i}(M, g)$$

is a Fredholm operator on its domain endowed with the graph norm and its index satisfies

$$\text{ind}((d_m + d_m^*)_{ev}) = \chi_{m \rightarrow M}(M, g) \quad (48)$$

*Proof.* By the fact that  $(L^2 \Omega^i(M, g), d_{m,i})$  is a Fredholm complex it follows that  $d_m + d_m^*$  is a Fredholm operator on its domain endowed with graph norm. Now if we define  $(d_m + d_m^*)_{odd}$  analogously to  $(d_m + d_m^*)_{ev}$ , then it is clear that  $\mathcal{D}(d_m + d_m^*) = \mathcal{D}((d_m + d_m^*)_{ev}) \oplus \mathcal{D}((d_m + d_m^*)_{odd})$ , that  $\text{Ker}(d_m + d_m^*) = \text{Ker}((d_m + d_m^*)_{ev}) \oplus \text{Ker}((d_m + d_m^*)_{odd})$  and that  $\text{ran}(d_m + d_m^*) = \text{ran}((d_m + d_m^*)_{ev}) \oplus \text{ran}((d_m + d_m^*)_{odd})$ . This implies immediately that also  $(d_m + d_m^*)_{ev}$  is a Fredholm operator on its domain endowed with the graph norm. Finally (48) is an easy consequence of the Hodge theorem stated in corollary 4.  $\square$

## 2.2 A topological obstruction to the existence of riemannian metric with finite $L^2$ cohomology

Now we want to show another application of the vector spaces  $\overline{H}_{2,m \rightarrow M}^i(M, g)$ . Consider again the complex  $(\Omega_c^*(M), d_*)$ . We will call a **closed extension** of  $(\Omega_c^*(M), d_*)$  any Hilbert complex  $(L^2 \Omega^i(M, g), D_i)$  where  $D_i : L^2 \Omega^i(M, g) \rightarrow L^2 \Omega^{i+1}(M, g)$  is a densely defined, closed operator which extends  $d_i : \Omega_c^i(M, g) \rightarrow \Omega_c^{i+1}(M, g)$  and such that the action of  $D_i$  on  $\mathcal{D}(D_i)$ , its domain, coincides with the action of  $d_i$  on  $\mathcal{D}_i$  in a distributional way. Obviously for



every closed extension of  $(\Omega_c^*(M), d_*)$  we have  $(L^2\Omega^*(M, g), d_{min,*}) \subseteq (L^2\Omega^*(M, g), D_i) \subseteq (L^2\Omega^*(M, g), d_{max,*})$ . We will label with  $\overline{H}_{2,D_*}^i(M, g)$ ,  $H_{2,D_*}^i(M, g)$  respectively the reduced cohomology and the cohomology groups of  $(L^2\Omega^i(M, g), D_i)$  and with  $\mathcal{H}_{D_*}^i(M, g)$  its Hodge cohomology groups. Moreover if  $(L^2\Omega^*(M, g), D'_i)$  is another closed extension of  $(\Omega_c^*(M), d_*)$  such that  $(L^2\Omega^*(M, g), D_i) \subseteq (L^2\Omega^*(M, g), D'_i)$  we will label with  $H_{2,D \rightarrow D'}^i(M, g)$ ,  $\overline{H}_{2,D \rightarrow D'}^i(M, g)$  respectively the image of the cohomology groups, reduced cohomology groups, of the complex  $(L^2\Omega^*(M, g), D_i)$  into the cohomology groups, reduced cohomology groups, of the complex  $(L^2\Omega^*(M, g), D'_i)$  induced by the natural inclusion of complexes.

Before we proceed we need the following propositions.

**Proposition 13.** *Let  $(M, g)$  be an incomplete and oriented riemannian manifold of dimension  $m$ . For each  $i = 0, \dots, m$  consider  $\mathcal{D}(d_{max,i})$ . Let  $\omega \in \mathcal{D}(d_{max,i})$ . Then there exists a sequence of smooth forms  $\{\omega_j\}_{j \in \mathbb{N}} \subset \Omega^i(M) \cap L^2\Omega^i(M, g)$  such that :*

1.  $d_i\omega_j \in L^2\Omega^{i+1}(M, g)$ .
2.  $\omega_j \rightarrow \omega$  in  $L^2\Omega^i(M, g)$ .
3.  $d_i\omega_j \rightarrow d_{max,i}\omega$  in  $L^2\Omega^{i+1}(M, g)$ .

*Proof.* See [8] pag 93. □

The next proposition is a variation of a result of de Rham, see [9] theorem 24.

**Proposition 14.** *Let  $(M, g)$  an incomplete and oriented riemannian manifold of dimension  $m$ . For each  $i = 0, \dots, m$  consider  $\mathcal{D}(d_{max,i})$ . Let  $\omega \in \overline{\text{ran}(d_{max,i})}$  such that  $\omega$  is smooth. Then there exist  $\eta \in \Omega^i(M)$  such that  $d_i\eta = \omega$ .*

*Proof.* By Poincaré duality between de Rham cohomology and compactly supported de Rham cohomology on an open and oriented manifold we know that it sufficient to show that

$$\int_M \omega \wedge \phi = 0$$

for each closed and compactly supported  $n - i$ -form  $\phi$  to get that  $\omega$  is an exact  $i$ -form in the smooth de Rham complex. Now, by proposition 13, we know that there exists a sequence of smooth  $i$ -forms  $\{\omega_j\}_{j \in \mathbb{N}}$  such that  $d_i\omega_j \rightarrow \omega$  in  $L^2\Omega^{i+1}(M, g)$ . Then:

$$\begin{aligned} \int_M \omega \wedge \phi &= \int_M \left( \lim_{j \rightarrow \infty} d_i\omega_j \right) \wedge \phi = \lim_{j \rightarrow \infty} \int_M d_i\omega_j \wedge \phi = \pm \lim_{j \rightarrow \infty} \int_M d_i\omega_j \wedge (*\phi) = \\ &= \pm \lim_{j \rightarrow \infty} \langle d_i\omega_j, *\phi \rangle_{L^2(M, g)} = \pm \lim_{j \rightarrow \infty} \langle \omega_j, \delta_{i-1}(*\phi) \rangle_{L^2(M, g)} = 0. \end{aligned}$$

So the proposition is proved. □

**Proposition 15.** *Let  $(L^2\Omega^i(M, g), D_i)$  any closed extension of  $(\Omega_c^*(M), d_*)$  where  $(M, g)$  is an incomplete oriented riemannian manifold. Then every cohomology class in  $\overline{H}_{2,D_*}^i(M, g)$  has a smooth representative. The same conclusion holds for every cohomology class in  $H_{2,D_*}^i(M, g)$ .*

*Proof.* By (11) we know that every cohomology class in  $\overline{H}_{2,D_*}^i(M, g)$  has a representative in  $\mathcal{H}_{D_*}^i(M, g)$ . Now, by elliptic regularity (see for example de Rham book [9]), it follows that every element in  $\mathcal{H}_{D_*}^i(M, g)$  is smooth. Now if we look at proposition 4, elliptic regularity tells us again that every element in  $\mathcal{D}^\infty(L^2\Omega^i(M, g))$  is smooth. Therefore from this it follows immediately the statement for  $H_{2,D_*}^i(M, g)$ . □

From the above propositions 14 and 15 it follows that that there exists a well defined map from  $\overline{H}_{2,D_*}^i(M, g)$ , respectively from  $H_{2,D_*}^i(M, g)$ , to the ordinary de Rham cohomology of  $M$  which assigns to each cohomology class  $[\omega] \in \overline{H}_{2,D_*}^i(M, g)$ , respectively  $[\omega] \in H_{2,D_*}^i(M, g)$ , the cohomology class in  $H_{dR}^i(M)$  given by the smooth representatives of  $[\omega]$ . By proposition 14 this cohomology class in  $H_{dR}^i(M)$  does not depend from the choice of the smooth representative of

$[\omega]$  and therefore this map is well defined.

We will label these maps:

$$s_{D,i}^* : H_{2,D_*}^i(M, g) \longrightarrow H_{dR}^i(M) \text{ in the non reduced case} \quad (49)$$

and

$$s_{r,D,i}^* : \overline{H}_{2,D_*}^i(M, g) \longrightarrow H_{dR}^i(M) \text{ in the reduced case} \quad (50)$$

In particular for the maximal and minimal extension we will label these maps:

$$s_{M,i}^* : H_{2,max}^i(M, g) \longrightarrow H_{dR}^i(M) \text{ in the non reduced case} \quad (51)$$

and

$$s_{r,M,i}^* : \overline{H}_{2,max}^i(M, g) \longrightarrow H_{dR}^i(M) \text{ in the reduced case} \quad (52)$$

and analogously for the minimal extension

$$s_{m,i}^* : H_{2,min}^i(M, g) \longrightarrow H_{dR}^i(M) \text{ in the non reduced case} \quad (53)$$

and

$$s_{r,m,i}^* : \overline{H}_{2,min}^i(M, g) \longrightarrow H_{dR}^i(M) \text{ in the reduced case} \quad (54)$$

Now we are ready to state the following proposition:

**Proposition 16.** *Let  $(M, g)$  be an open, oriented and incomplete riemannian manifold. Let  $(L^2\Omega^i(M, g), D_{a,*}), (L^2\Omega^i(M, g), D_{b,*})$  be two closed extension of  $(\Omega_c^*(M), d_*)$  such that*

$$(L^2\Omega^*(M, g), d_{min,*}) \subseteq (L^2\Omega^*(M, g), D_{a,*}) \subseteq (L^2\Omega^*(M, g), D_{b,*}) \subseteq (L^2\Omega^*(M, g), d_{max,*}). \quad (55)$$

Then the two following diagrams commute:

$$\begin{array}{ccccc} H_c^i(M) & \longrightarrow & H_{dR}^i(M) & & H_c^i(M) \longrightarrow H_{dR}^i(M) \\ \downarrow & & \uparrow s_{M,i}^* & & \downarrow & & \uparrow s_{r,M,i}^* \\ H_{2,min}^i(M, g) & \longrightarrow & H_{2,max}^i(M, g) & & \overline{H}_{2,min}^i(M, g) \longrightarrow \overline{H}_{2,max}^i(M, g) \\ \downarrow & & \uparrow & & \downarrow & & \uparrow \\ H_{2,D_{a,*}}^i(M, g) & \longrightarrow & H_{2,D_{B,*}}^i(M, g) & & \overline{H}_{2,D_{a,*}}^i(M, g) \longrightarrow \overline{H}_{2,D_{B,*}}^i(M, g) \end{array} \quad (56)$$

where all the above arrows without label are the natural maps between cohomology, respectively reduced cohomology groups, induced by the natural inclusion of the relative complexes.

*Proof.* It is clear that both the two following diagrams commute:

$$\begin{array}{ccccc} H_c^i(M) & & & & H_c^i(M) \\ \downarrow & \searrow & & \searrow & \downarrow \\ H_{2,min}^i(M, g) & \longrightarrow & H_{2,max}^i(M, g) & & \overline{H}_{2,min}^i(M, g) \longrightarrow \overline{H}_{2,max}^i(M, g) \\ \downarrow & & \uparrow & & \downarrow & & \uparrow \\ H_{2,D_{a,*}}^i(M, g) & \longrightarrow & H_{2,D_{B,*}}^i(M, g) & & \overline{H}_{2,D_{a,*}}^i(M, g) \longrightarrow \overline{H}_{2,D_{B,*}}^i(M, g) \end{array}$$

So, to complete the proof, we have to show that the two following diagrams are both commutative:

$$\begin{array}{ccc} H_c^i(M) & & H_c^i(M) \\ \downarrow & \searrow & \downarrow \\ H_{2,max}^i(M, g) & \xrightarrow{s_{M,i}^*} & H_{dR}^i(M) \quad \overline{H}_{2,max}^i(M, g) \xrightarrow{s_{r,M,i}^*} H_{dR}^i(M) \end{array}$$

To prove this it is enough to show that given a cohomology class  $[\omega] \in H_{2,max}^i(M, g)$ , respectively  $[\omega] \in \overline{H}_{2,max}^i(M, g)$ , such that  $\omega$  is closed, smooth and with compact support, if  $[\omega] = 0$  in  $H_{2,max}^i(M, g)$  or in  $\overline{H}_{2,max}^i(M, g)$  then also  $s_{M,i}^*(\omega) = 0$ , respectively  $s_{r,M,i}^*(\omega) = 0$ , that is the cohomology class of  $\omega$  in  $H_{dR}^i(M)$  is null. This last statement follows immediately from proposition 14.  $\square$

Using the previous proposition we get the following corollary in which the first statement extends a result of Anderson, see [2], to the case of an incomplete riemannian metric both for the reduced and the unreduced  $L^2$  cohomology groups.

**Corollary 5.** *Let  $(M, g)$  as in the previous proposition. Then, for each  $j = 0, \dots, \dim M$ , there are injective maps:*

$$\text{im}(H_c^j(M) \rightarrow H_{dR}^j(M)) \longrightarrow \overline{H}_{2,m \rightarrow M}^j(M, g) \longrightarrow \overline{H}_{2,D_a \rightarrow D_b}^j(M, g) \quad (57)$$

$$\text{im}(H_c^j(M) \rightarrow H_{dR}^j(M)) \longrightarrow H_{2,m \rightarrow M}^j(M, g) \longrightarrow H_{2,D_a \rightarrow D_b}^j(M, g) \quad (58)$$

Moreover if  $H_c^i(M) \rightarrow H_{dR}^i(M)$  is injective then

$$H_c^i(M) \rightarrow H_{2,m \rightarrow M}^i(M, g), \quad H_c^i(M) \rightarrow \overline{H}_{2,m \rightarrow M}^i(M, g) \quad (59)$$

are injective and therefore for each closed extension  $(L^2\Omega^*(M, g), D_*)$  also the following maps are injective:

$$H_c^i(M) \rightarrow H_{2,D}^i(M, g), \quad H_c^i(M) \rightarrow \overline{H}_{2,D}^i(M, g) \quad (60)$$

*Proof.* It is an immediate consequence of the previous proposition.  $\square$

Now we give other three corollaries of proposition 16. In particular the third corollary shows that it could exist a **topological obstruction** to the existence of a riemannian metric on  $g$  with certain analytic properties.

**Corollary 6.** *Let  $M$  an open manifold such that for some  $j$   $\text{im}(H_c^j(M) \xrightarrow{i_j^*} H_{dR}^j(M))$  is non trivial. Then for every riemannian metric  $g$  on  $M$  and for every pair of closed extensions  $(L^2\Omega^*(M, g), D_{a,*})$ ,  $(L^2\Omega^*(M, g), D_{b,*})$  such that  $(L^2\Omega^*(M, g), D_{a,*}) \subseteq (L^2\Omega^*(M, g), D_{b,*})$  we have that for the same  $j$  both vector spaces*

$$H_{2,D_a \rightarrow D_b}^j(M, g), \quad \overline{H}_{2,D_a \rightarrow D_b}^j(M, g)$$

are non trivial. In particular this implies that for the same  $j$  the following four vector spaces are non trivial:

$$H_{2,D_a}^j(M, g), \quad H_{2,D_b}^j(M, g), \quad \overline{H}_{2,D_a}^j(M, g), \quad \overline{H}_{2,D_b}^j(M, g).$$

**Corollary 7.** *Let  $(M, g)$  an open, oriented and incomplete riemannian manifold. Suppose that there exists a pair of closed extension  $(L^2\Omega^*(M, g), D_{a,*})$ ,  $(L^2\Omega^*(M, g), D_{b,*})$  of  $(\Omega_c^*(M), d_*)$  such that are both weak Fredholm and  $(L^2\Omega^*(M, g), D_{a,*}) \subseteq (L^2\Omega^*(M, g), D_{b,*})$ .*

*Then  $\text{im}(H_c^j(M) \xrightarrow{i_j^*} H_{dR}^j(M))$  is finite dimensional and we have*

$$\dim(\text{im}(H_c^j(M) \xrightarrow{i_j^*} H_{dR}^j(M))) \leq \dim \overline{H}_{2,D_a}^j(M, g) \quad (61)$$

$$\dim(\text{im}(H_c^j(M) \xrightarrow{i_j^*} H_{dR}^j(M))) \leq \dim \overline{H}_{2,D_b}^j(M, g) \quad (62)$$

*In particular if one of the two complexes  $(L^2\Omega^*(M, g), d_{\max/\min,*})$  is weak Fredholm then also the other one is weak Fredholm and for each  $j = 0, \dots, m$  we have:*

$$\dim(\text{im}(H_c^j(M) \xrightarrow{i_j^*} H_{dR}^j(M))) \leq \dim \overline{H}_{2,max}^j(M, g) \quad (63)$$

$$\dim(\text{im}(H_c^j(M) \xrightarrow{i_j^*} H_{dR}^j(M))) \leq \dim \overline{H}_{2,min}^j(M, g). \quad (64)$$

Finally if one of the two complexes  $(L^2\Omega^*(M, g), d_{\max/\min,*})$  is Fredholm then for each  $j = 0, \dots, m$  we have:

$$\dim(\text{im}(H_c^j(M) \xrightarrow{i_j^*} H_{dR}^j(M))) \leq \dim H_{2,\max}^j(M, g) \quad (65)$$

$$\dim(\text{im}(H_c^j(M) \xrightarrow{i_j^*} H_{dR}^j(M))) \leq \dim H_{2,min}^j(M, g). \quad (66)$$

*Proof.* It is an immediate consequence of corollary 5.  $\square$

**Corollary 8.** *Let  $M$  an open, oriented and incomplete riemannian manifold where  $m = \dim(M)$ . Suppose that there exists an  $j \in \{0, \dots, m\}$  such that  $\text{im}(H_c^j(M) \xrightarrow{i_j^*} H_{dR}^j(M))$  is infinite dimensional. Then  $M$  does not admit any riemannian metrics  $g$  (complete or incomplete) such that  $g$  implies the existence of a closed extension  $(L^2\Omega^*(M, g), D_*)$  of  $(\Omega_c^*(M), d_*)$  with one of the following properties for the same  $j$ :*

1.  $\overline{H}_{2,D_*}^j(M, g)$  or  $\overline{H}_{2,D_*}^{m-j}(M, g)$  is finite dimensional.
2.  $H_{2,D_*}^j(M, g)$  or  $H_{2,D_*}^{m-j}(M, g)$  is finite dimensional.
3.  $D_j^* \circ D_j + D_{j-1} \circ D_{j-1}^*$  on its domain (as defined in (9)) endowed with the graph norm is a Fredholm operator.

Moreover  $M$  does not admit any riemannian metric  $g$  such that:

1.  $\Delta_{\max,j}$ , the maximal closed extension of  $\Delta_j : \Omega_c^j(M) \rightarrow \Omega_c^j(M)$ , has finite dimensional nullspace.
2.  $\Delta_{\min,j}$ , the minimal closed extension of  $\Delta_j : \Omega_c^j(M) \rightarrow \Omega_c^j(M)$ , satisfies  $\dim(\text{ran}(\Delta_{\min,j})^\perp) < \infty$ .

*Proof.* The first two points are immediate consequence of corollary 5 and theorem 11. The third point follows immediately by (10) and (11).

Finally, for the last two points, if  $\text{Ker}(\Delta_{\max,j})$  is finite dimensional then all the other closed extensions of  $\Delta_j : \Omega_c^j(M) \rightarrow \Omega_c^j(M)$  have finite dimensional nullspace. So we can apply the third point to get the conclusion. Finally if we consider  $\Delta_{\min,j}$  then we have  $\Delta_{\min,j}^* = \Delta_{\max,j}$ . So if  $\dim(\text{ran}(\Delta_{\min,j})^\perp) < \infty$  then  $\text{Ker}(\Delta_{\max,j})$  is finite dimensional. Now by the previous point we can get the conclusion.  $\square$

### 2.3 $L^2$ and topological signature on an open oriented and incomplete riemannian manifold.

The aim of this subsection is to show that if  $(M, g)$  is an open oriented and incomplete riemannian manifold such that for  $i = 2k$   $\overline{H}_{2,m \rightarrow M}^i(M, g)$  is finite dimensional, where  $4k = \dim M$ , then we can define over  $M$  a  $L^2$  signature and a topological signature. The first step is to show that using the wedge product we can construct a well defined and non degenerate pairing between  $\overline{H}_{2,m \rightarrow M}^i(M, g)$  and  $\overline{H}_{2,m \rightarrow M}^{n-i}(M, g)$  where  $n = \dim M$ . In fact any cohomology class  $[\omega] \in \overline{H}_{2,m \rightarrow M}^i(M, g)$  is a cohomology class in  $\overline{H}_{2,\max}^i(M, g)$  which admits a representative in  $\text{Ker}(d_{\min,i})$ . So we can define:

$$\overline{H}_{2,m \rightarrow M}^i(M, g) \times \overline{H}_{2,m \rightarrow M}^{n-i}(M, g) \longrightarrow \mathbb{R}, ([\eta], [\omega]) \mapsto \int_M \eta \wedge \omega \quad (67)$$

where  $\omega \in \text{Ker}(d_{\min,i})$  and  $\eta \in \text{Ker}(d_{\min,n-i})$

**Proposition 17.** *Let  $(M, g)$  be an open, oriented and incomplete riemannian manifold of dimension  $n$ . Then (67) is a well defined and non degenerate pairing and therefore when the vector spaces  $\overline{H}_{2, m \rightarrow M}^i(M, g)$   $i = 0, \dots, n$  are finite dimensional it induces an isomorphism between*

$$\overline{H}_{2, m \rightarrow M}^i(M, g) \text{ and } (\overline{H}_{2, m \rightarrow M}^{n-i}(M, g))^*.$$

*Proof.* The first step is to show that (67) is well defined. Let  $\eta', \omega'$  other two forms such that  $[\eta] = [\eta']$  in  $\overline{H}_{2, m \rightarrow M}^i(M, g)$ ,  $[\omega] = [\omega']$  in  $\overline{H}_{2, m \rightarrow M}^{n-i}(M, g)$  and that  $\omega' \in \text{Ker}(d_{\min, i})$ ,  $\eta' \in \text{Ker}(d_{\min, n-i})$ . Then there exist  $\alpha \in \overline{d_{\max, i-1}} \cap \mathcal{D}(d_{\min, i})$  and  $\beta \in \overline{d_{\max, n-i-1}} \cap \mathcal{D}(d_{\min, n-i})$  such that  $\eta = \eta' + \alpha$  and  $\omega = \omega' + \beta$ . Therefore

$$\int_M \eta \wedge \omega = \int_M (\eta' + \alpha) \wedge (\omega' + \beta) = \int_M \eta' \wedge \omega' + \int_M \eta' \wedge \beta + \int_M \alpha \wedge \omega' + \int_M \alpha \wedge \beta$$

Now

$$\int_M \eta' \wedge \beta = \pm \int_M \langle \eta', * \beta \rangle d\text{vol}_M = \langle \eta', * \beta \rangle_{L^2(M, g)} = 0$$

because  $\text{Ker}(d_{\min, i})^\perp = \overline{\text{ran}(d_{\max, i})}$ . In the same way:

$$\int_M \alpha \wedge \beta = \pm \int_M \langle \alpha, * \beta \rangle d\text{vol}_M = \langle \alpha, * \beta \rangle_{L^2(M, g)} = 0.$$

Finally

$$\int_M \alpha \wedge \omega' = \pm \int_M \langle \alpha, * \omega' \rangle d\text{vol}_M = \langle \alpha, * \omega' \rangle_{L^2(M, g)} = 0$$

because  $\text{Ker}(d_{\min, i-1})^\perp = \overline{\text{ran}(d_{\max, i-1})}$ . So we can conclude that (67) is well defined. Now fix  $[\eta] \in \overline{H}_{2, m \rightarrow M}^i(M, g)$  and suppose that for each  $[\omega] \in \overline{H}_{2, m \rightarrow M}^{n-i}(M, g)$  the pairing (67) vanishes. Then this means that for each  $\omega \in \text{Ker}(d_{\min, n-i})$  we have  $\int_M \eta \wedge \omega = 0$ . We also know that  $\int_M \eta \wedge \omega = \langle \eta, * \omega \rangle_{L^2(M, g)}$  and that  $*(\text{Ker}(d_{\min, n-i})) = \text{Ker}(d_{\min, i-1})$ . So by the fact that  $(\text{Ker}(d_{\min, i-1}))^\perp = \overline{\text{ran}(d_{\max, i-1})}$  we obtain that  $[\eta] = 0$ . In the same way if  $[\omega] \in \overline{H}_{2, m \rightarrow M}^{n-i}(M, g)$  is such that for each  $[\eta] \in \overline{H}_{2, m \rightarrow M}^i(M, g)$  the pairing (67) vanishes then we know that for each  $\eta \in \text{Ker}(d_{\max, i})$  we have  $\int_M \eta \wedge \omega = 0$ . But we know that  $\int_M \eta \wedge \omega = \langle \eta, * \omega \rangle_{L^2(M, g)}$ . So by the fact that  $*(\overline{\text{ran}(d_{\max, n-i-1})}) = \overline{\text{ran}(d_{\max, i})}$  and that  $(\text{Ker}(d_{\min, i}))^\perp = \overline{\text{ran}(d_{\max, i})}$  we obtain that  $[\omega] = 0$ .

So we can conclude that the pairing (67) is well defined and non degenerate and therefore when the vector spaces  $\overline{H}_{2, m \rightarrow M}^i(M, g)$   $i = 0, \dots, n$  are finite dimensional it induces an isomorphism between  $\overline{H}_{2, m \rightarrow M}^i(M, g)$  and  $(\overline{H}_{2, m \rightarrow M}^{n-i}(M, g))^*$ .  $\square$

**Remark 4.** *We can look at this proposition as an alternative statement (and proof) of theorem 11.*

We have the following immediate corollary:

**Corollary 9.** *Let  $(M, g)$  an open, oriented and incomplete riemannian manifold of dimension  $4n$ . Then on  $\overline{H}_{2, m \rightarrow M}^{2n}(M, g)$  the pairing (67) is a symmetric bilinear form.*

We can now state the following definition:

**Definition 6.** *Let  $(M, g)$  an open and oriented riemannian manifold of dimension  $4n$  such that, for  $i = 2n$ ,  $\overline{H}_{2, m \rightarrow M}^{2n}(M, g)$  is finite dimensional. Then we define the  $L^2$  signature of  $(M, g)$  and we label it  $\sigma_2(M, g)$  as the signature of the pairing (67) on  $\overline{H}_{2, m \rightarrow M}^{2n}(M, g)$ .*

Consider now the sequence of vector spaces  $\text{im}(H_c^i(M) \rightarrow H_{dR}^i(M))$   $i = 0, \dots, \dim M$ . A cohomology class in  $\text{im}(H_c^i(M) \rightarrow H_{dR}^i(M))$  is a cohomology class in  $H_{dR}^i(M)$  which admits as representative a smooth and closed form with compact support. So in a similar way to the previous case we can define:

$$\text{im}(H_c^i(M) \rightarrow H_{dR}^i(M)) \times \text{im}(H_c^{n-i}(M) \rightarrow H_{dR}^{n-i}(M)) \longrightarrow \mathbb{R}, ([\eta], [\omega]) \mapsto \int_M \eta \wedge \omega \quad (68)$$

where  $\omega$  is a  $i$ -form closed with compact support and in the same way  $\eta$  is a closed  $n-i$ -form with compact support. Now by Poincaré duality for open and oriented manifolds it follows easily that this pairing is well defined and non degenerate. So we can conclude that, if for each  $i = 0, \dots, \dim M$   $\text{im}(H_c^i(M) \rightarrow H_{dR}^i(M))$  is finite dimensional, then the pairing 68 induces an isomorphism between  $\text{im}(H_c^i(M) \rightarrow H_{dR}^i(M))$  and  $\text{im}(H_c^{n-i}(M) \rightarrow H_{dR}^{n-i}(M))^*$ . Moreover it is clear that when  $\dim M = 4n$  then, for  $i = 2n$ , (68) is a symmetric bilinear form. This implies that when  $\dim M = 4n$  it is possible to define a signature on  $M$ , which is topological by de Rham isomorphism theorem, taking the signature of the pairing (68) for  $i = 2n$ . This leads us to state the next proposition:

**Proposition 18.** *Let  $(M, g)$  be an open, oriented and incomplete riemannian manifold of dimension  $4n$ . If  $(M, g)$  admits the  $L^2$  signature  $\sigma_2(M, g)$  of definition 6 then it admits also a topological signature defined as the signature of the pairing (68) on  $\text{im}(H_c^{2n}(M) \rightarrow H_{dR}^{2n}(M))$ .*

*Proof.* If  $M$  admits the signature  $\sigma_2(M, g)$  then, by definition 6, we know that  $\overline{H}_{2, m \rightarrow M}^{2n}(M, g)$  is finite dimensional. Now, by corollary 5, we know that also  $\text{im}(H_c^{2n}(M) \rightarrow H^{2n}(M))$  is finite dimensional and so the pairing 68 admits a signature.  $\square$

Moreover in the next section we will see that, on a class of open, incomplete and oriented riemannian manifold, the  $L^2$  signature of definition 6 has a topological meaning.

### 3 Topological Applications

The aim of this section is to exhibit some topological and geometrical applications of the previous results. In the first part we show some application to the intersection cohomology with general perversity of a compact and smoothly stratified pseudomanifold. In the last part we exhibit some examples for which corollary 8 applies.

To get the paper as self contained as possible we will recall briefly the definitions of smoothly stratified pseudomanifold with a Thom-Mather stratification, quasi edge metric and intersection cohomology with general perversity.

#### 3.1 A brief reminder on (smoothly) stratified pseudomanifolds and intersection cohomology

We start this subsection by recalling the notions of a smoothly stratified pseudomanifold with a Thom-Mather stratification. For the more general (and simple) definition of stratified pseudomanifold we refer to [3] and [19].

**Definition 7.** *A smoothly stratified pseudomanifold  $X$  with a Thom-Mather stratification is a metrizable, locally compact, second countable space which admits a locally finite decomposition into a union of locally closed strata  $\mathfrak{S} = \{Y_\alpha\}$ , where each  $Y_\alpha$  is a smooth, open and connected manifold, with dimension depending on the index  $\alpha$ . We assume the following:*

1. *If  $Y_\alpha, Y_\beta \in \mathfrak{S}$  and  $Y_\alpha \cap \overline{Y_\beta} \neq \emptyset$  then  $Y_\alpha \subset \overline{Y_\beta}$*
2. *Each stratum  $Y$  is endowed with a set of control data  $T_Y, \pi_Y$  and  $\rho_Y$ ; here  $T_Y$  is a neighborhood of  $Y$  in  $X$  which retracts onto  $Y$ ,  $\pi_Y : T_Y \rightarrow Y$  is a fixed continuous retraction and  $\rho_Y : T_Y \rightarrow [0, 2)$  is a proper radial function in this tubular neighborhood such that  $\rho_Y^{-1}(0) = Y$ . Furthermore, we require that if  $Z \in \mathfrak{S}$  and  $Z \cap T_Y \neq \emptyset$  then  $(\pi_Y, \rho_Y) : T_Y \cap Z \rightarrow Y \times [0, 2)$  is a proper differentiable submersion.*
3. *If  $W, Y, Z \in \mathfrak{S}$ , and if  $p \in T_Y \cap T_Z \cap W$  and  $\pi_Z(p) \in T_Y \cap Z$  then  $\pi_Y(\pi_Z(p)) = \pi_Y(p)$  and  $\rho_Y(\pi_Z(p)) = \rho_Y(p)$ .*
4. *If  $Y, Z \in \mathfrak{S}$ , then  $Y \cap \overline{Z} \neq \emptyset \Leftrightarrow T_Y \cap Z \neq \emptyset$ ,  $T_Y \cap T_Z \neq \emptyset \Leftrightarrow Y \subset \overline{Z}, Y = Z$  or  $Z \subset \overline{Y}$ .*
5. *For each  $Y \in \mathfrak{S}$ , the restriction  $\pi_Y : T_Y \rightarrow Y$  is a locally trivial fibration with fibre the cone  $C(L_Y)$  over some other stratified space  $L_Y$  (called the link over  $Y$ ), with atlas  $\mathcal{U}_Y = \{(\phi, \mathcal{U})\}$  where each  $\phi$  is a trivialization  $\pi_Y^{-1}(U) \rightarrow U \times C(L_Y)$ , and the transition functions are stratified isomorphisms which preserve the rays of each conic fibre as well as the radial variable  $\rho_Y$  itself, hence are suspensions of isomorphisms of each link  $L_Y$  which vary smoothly with the variable  $y \in U$ .*

6. For each  $j$  let  $X_j$  be the union of all strata of dimension less or equal than  $j$ , then

$$X - X_{n-1} \text{ is dense in } X$$

The **depth** of a stratum  $Y$  is largest integer  $k$  such that there is a chain of strata  $Y = Y_k, \dots, Y_0$  such that  $Y_j \subset \overline{Y_{j-1}}$  for  $i \leq j \leq k$ . A stratum of maximal depth is always a closed subset of  $X$ . The maximal depth of any stratum in  $X$  is called the **depth of  $X$**  as stratified spaces. Consider the filtration

$$X = X_n \supset X_{n-1} \supset X_{n-2} \supset X_{n-3} \supset \dots \supset X_0 \quad (69)$$

We refer to the open subset  $X - X_{n-1}$  of a stratified pseudomanifold  $X$  as its regular set, and the union of all other strata as the singular set,

$$\text{reg}(X) := X - \text{sing}(X) \text{ where } \text{sing}(X) := \bigcup_{Y \in \mathfrak{S}, \text{depth} Y > 0} Y.$$

For more details and properties we refer to [1].

Now we take from [4] the following definition and result. Before giving the definition we recall that two riemannian metrics  $g, h$  on a smooth manifold  $M$  are **quasi-isometric** if there are constants  $c_1, c_2$  such that  $c_1 h \leq g \leq c_2 h$ .

**Definition 8.** Let  $X$  be a smoothly stratified pseudomanifold with a Thom-Mather stratification and let  $g$  a riemannian metric on  $\text{reg}(X)$ . We call  $g$  a **quasi edge metric with weights** if it satisfies the following properties:

1. Take any stratum  $Y$  of  $X$ ; by definition 7 for each  $q \in Y$  there exist an open neighborhood  $U$  of  $q$  in  $Y$  such that  $\phi : \pi_Y^{-1}(U) \rightarrow U \times C(L_Y)$  is a stratified isomorphism; in particular  $\phi : \pi_Y^{-1}(U) \cap \text{reg}(X) \rightarrow U \times \text{reg}(C(L_Y))$  is a diffeomorphism. Then, for each  $q \in Y$ , there exists one of these trivializations  $(\phi, U)$  such that  $g$  restricted on  $\pi_Y^{-1}(U) \cap \text{reg}(X)$  satisfies the following properties:

$$(\phi^{-1})^*(g|_{\pi_Y^{-1}(U) \cap \text{reg}(X)}) \cong dr \otimes dr + h_U + r^{2c} g_{L_Y} \quad (70)$$

where  $h_U$  is a riemannian metric defined over  $U$ ,  $c \in \mathbb{R}$  and  $c > 0$ ,  $g_{L_Y}$  is a riemannian metric on  $\text{reg}(L_Y)$ ,  $dr \otimes dr + h_U + r^{2c} g_{L_Y}$  is a riemannian metric of product type on  $U \times \text{reg}(C(L_Y))$  and with  $\cong$  we mean **quasi-isometric**.

2. If  $p$  and  $q$  lie in the same stratum  $Y$  then in (70) there is the same weight. We label it  $c_Y$ .

**Remark 5.** Implicit in the above definition is the fact that if the codimension of  $Y$  is 1 then  $L_Y$  is just a point and therefore  $(\phi^{-1})^*(g|_{\pi_Y^{-1}(U) \cap \text{reg}(X)}) \cong dr \otimes dr + h_U$ .

We refer to [4] for more comments about the above definitions, for some properties about metrics of this kind and for the proof of the following proposition.

**Proposition 19.** Let  $X$  be a smoothly stratified pseudomanifold with a Thom-Mather stratification  $\mathfrak{X}$ . For any stratum  $Y \subset X$  fix a positive real number  $c_Y$ . Then there exists a quasi edge metric with weights  $g$  on  $\text{reg}(X)$  having the numbers  $\{c_Y\}_{Y \in \mathfrak{X}}$  as weights.

Now we need to recall briefly the notion of intersection homology with general perversities. Intersection homology is a deep and rich field of algebraic topology founded by Mark Goresky and Robert MacPherson at the end of seventies. From the first two fundamental papers, [16] and [17], there have been several developments and the original theory has been extended in many directions. Our intention now is to recall briefly the extension of intersection homology given by Greg Friedman in [11]. For the original theory introduced by Goresky and MacPherson and also for more topological property about stratified pseudomanifolds we refer to [16], [17], [3] and [19].

**Definition 9.** Let  $X$  be a compact and oriented stratified pseudomanifold of dimension  $n$ . A general perversity on  $X$  is any function

$$p : \{\text{Singular Strata of } X\} \rightarrow \mathbb{Z}. \quad (71)$$

The dual perversity of  $p$ , usually labelled  $q$ , is the general perversity defined in this way

$$q = t - p \quad (72)$$

where  $t$  is the top perversity that is, given a singular stratum  $Z$  of  $X$ ,  $t(Z) = \text{cod}(Z) - 2$ .

**Example 1.** The upper middle perversity

$$\overline{m} : \{\text{Singular Strata of } X\} \rightarrow \mathbb{Z}. \quad (73)$$

is defined in the following way:

$$\overline{m}(Y) = \left\lfloor \frac{\text{cod}(Y) - 1}{2} \right\rfloor$$

while the lower middle one is

$$t - \overline{m}.$$

Now we introduce the notion of  $p$ -**allowable** singular simplex : a singular  $i$ -simplex in  $X$ , i.e. a continuous map  $\sigma : \Delta_i \rightarrow X$ , is  $p$ -**allowable** if

$$\sigma^{-1}(Y) \subset \{(i - \text{cod}(Y) + p(Y)) - \text{skeleton of } \Delta_i\} \text{ for any singular stratum } Y \text{ of } X. \quad (74)$$

A key ingredient in this new theory is the notion of **homology with stratified coefficient system**.

**Definition 10.** Let  $X$  stratified pseudomanifold and let  $\mathcal{G}$  a local system on  $X - X_{n-1}$ . Then the stratified coefficient system  $\mathcal{G}_0$  is defined to consist of the pair of coefficient systems given by  $\mathcal{G}$  on  $X - X_{n-1}$  and the constant 0 system on  $X_{n-1}$  i.e. we think of  $\mathcal{G}_0$  as consisting of a locally constant fiber bundle  $\mathcal{G}_{X-X_{n-1}}$  over  $X - X_{n-1}$  with fiber  $G$  with the discrete topology together with the trivial bundle on  $X_{n-1}$  with the stalk 0.

Then a **coefficient**  $n$  of a singular simplex  $\sigma$  can be described by a lift of  $\sigma|_{\sigma^{-1}(X-X_{n-1})}$  to  $\mathcal{G}$  over  $X - X_{n-1}$  together with the trivial lift of  $\sigma|_{\sigma^{-1}(X_{n-1})}$  to the 0 system on  $X_{n-1}$ . A coefficient of a simplex  $\sigma$  is considered to be the 0 coefficient if it maps each points of  $\Delta$  to the 0 section of one of the coefficient systems. Note that if  $\sigma^{-1}(X - X_{n-1})$  is path-connected then a coefficient lift of  $\sigma$  to  $\mathcal{G}_0$  is completely determined by the lift at a single point of  $\sigma^{-1}(X - X_{n-1})$  by the lifting extension property for  $\mathcal{G}$ . The intersection homology chain complex  $(IP S_*(X, \mathcal{G}_0), \partial_*)$  are defined in the same way as  $IP S_*(X, G)$ , where  $G$  is any field, but replacing the coefficient of simplices with coefficient in  $\mathcal{G}_0$ . If  $n\sigma$  is a simplex  $\sigma$  with its coefficient  $n$ , its boundary is given by the usual formula  $\partial(n\sigma) = \sum_j (-1)^j (n \circ i_j)(\sigma \circ i_j)$  where  $i_j : \Delta_{i-1} \rightarrow \Delta_i$  is the  $j$ -face inclusion map. Here  $n \circ i_j$  should be interpreted as the restriction of  $n$  to the  $j$ th face of  $\sigma$ , restricting the lift to  $\mathcal{G}$  where possible and restricting to 0 otherwise. The basic idea behind the definition is that when we consider allowability of chains with respect to a perversity, simplices with support entirely in  $X_{n-1}$  should vanish and thus not be counted for allowability considerations. We recommend to the reader the references [10], [11] and [12] for a complete development of the subject.

Finally we conclude this subsection recalling from [4] the following definition and the next two theorems:

**Definition 11.** Let  $X$  be a smoothly stratified pseudomanifold with a Thom-Mather stratification and let  $g$  a quasi edge metric with weights on  $\text{reg}(X)$ . Then the general perversity  $p_g$  associated to  $g$  is:

$$p_g(Y) := Y \mapsto \left\lfloor \frac{l_Y}{2} + \frac{1}{2c_Y} \right\rfloor = \begin{cases} 0 & l_Y = 0 \\ \frac{l_Y}{2} + \left\lfloor \frac{1}{2c_Y} \right\rfloor & l_Y \text{ even and } l_Y \neq 0 \\ \frac{l_Y-1}{2} + \left\lfloor \frac{1}{2} + \frac{1}{2c_Y} \right\rfloor & l_Y \text{ odd} \end{cases} \quad (75)$$

where  $l_Y = \dim L_Y$  and, given any real and positive number  $x$ ,  $\lfloor x \rfloor$  is the greatest integer strictly less than  $x$ .



**Theorem 13.** *Let  $X$  be a compact and oriented smoothly stratified pseudomanifold of dimension  $n$  with a Thom-Mather stratification  $\mathfrak{X}$ . Let  $g$  be a quasi edge metric with weights on  $\text{reg}(X)$ , see definition 8. Let  $\mathcal{R}_0$  be the stratified coefficient system made of the pair of coefficient systems given by  $(X - X_{n-1}) \times \mathbb{R}$  over  $X - X_{n-1}$  where the fibers  $\mathbb{R}$  have the discrete topology and the constant 0 system on  $X_{n-1}$ . Let  $p_g$  be the general perversity associated to the metric  $g$ , see definition 11. Then, for all  $i = 0, \dots, n$ , the following isomorphisms holds:*

$$I^{q_g} H^i(X, \mathcal{R}_0) \cong H_{2, \max}^i(\text{reg}(X), g) \cong \mathcal{H}_{\text{abs}}^i(\text{reg}(X), g) \quad (76)$$

$$I^{p_g} H^i(X, \mathcal{R}_0) \cong H_{2, \min}^i(\text{reg}(X), g) \cong \mathcal{H}_{\text{rel}}^i(\text{reg}(X), g) \quad (77)$$

where  $q_g$  is the complementary perversity of  $p_g$ , that is,  $q_g = t - p_g$ ,  $t$  is the usual top perversity and  $\mathcal{H}_{\text{abs}/\text{rel}}^i(\text{reg}(X), g)$  are the Hodge cohomology groups defined in 32. In particular, for all  $i = 0, \dots, n$  the groups

$$H_{2, \max}^i(\text{reg}(X), g), H_{2, \min}^i(\text{reg}(X), g), \mathcal{H}_{\text{abs}}^i(\text{reg}(X), g), \mathcal{H}_{\text{rel}}^i(\text{reg}(X), g)$$

are all finite dimensional.

*Proof.* See [4] theorem 4. □

**Theorem 14.** *Let  $X$  be as in the previous theorem. Let  $p$  a general perversity in the sense of Friedman on  $X$ . If  $p$  satisfies the following conditions:*

$$\begin{cases} p \geq \overline{m} \\ p(Y) = 0 \quad \text{if } \text{cod}(Y) = 1 \end{cases} \quad (78)$$

then there exists  $g$ , a quasi edge metric with weights on  $\text{reg}(X)$ , such that

$$I^p H^i(X, \mathcal{R}_0) \cong H_{2, \min}^i(\text{reg}(X), g) \cong \mathcal{H}_{\text{rel}}^i(\text{reg}(X), g). \quad (79)$$

Conversely if  $p$  satisfies:

$$\begin{cases} p \leq \underline{m} \\ p(Y) = -1 \quad \text{if } \text{cod}(Y) = 1 \end{cases} \quad (80)$$

then, also in this case, there exists a quasi edge metric with weights  $h$  on  $\text{reg}(X)$  such that

$$I^p H^i(X, \mathcal{R}_0) \cong H_{2, \max}^i(\text{reg}(X), h) \cong \mathcal{H}_{\text{abs}}^i(\text{reg}(X), h). \quad (81)$$

*Proof.* See [4] theorem 5. □

### 3.2 Applications to the intersection cohomology

Now, after the previous reminder, we are ready to show some applications of the results of the previous sections.

**Proposition 20.** *Let  $X$  be a compact and oriented smoothly stratified pseudomanifold of dimension  $n$  with a Thom-Mather stratification  $\mathfrak{X}$ . Let  $g$  be a quasi edge metric with weights on  $\text{reg}(X)$ . Then*

$$H_{2, m \rightarrow M}^i(\text{reg}(X), g), \quad i = 0, \dots, n$$

is a finite sequence of finite dimensional vector spaces with Poincaré duality. Moreover proposition 10 and proposition 11 apply to this kind of riemannian manifolds.

*Proof.* By theorem 13 we know that both cohomology groups  $H_{2, \max/\min}^i(\text{reg}(X), g)$  are finite dimensional. This implies that in the following sequence  $H_{2, m \rightarrow M}^i(\text{reg}(X), g)$ ,  $i = 0, \dots, n$  each dimensional vector spaces is finite dimensional. In this way we are in position to apply theorem 11, proposition 10, proposition 11 and therefore the thesis follows. □

Now consider two general perversities  $p, q$  such that  $q \leq p$ . Then the complex associated to  $q$  is a subcomplex of that associated to  $p$  and therefore the inclusion  $i$  induces a maps between the intersection cohomology groups  $I^q H^i(X, \mathcal{R}_0)$  and  $I^p H^i(X, \mathcal{R}_0)$  that we call  $i^*$ . In analogy to the previous section we define for each  $j = 0, \dots, n$

$$I^{q \rightarrow p} H^j(X, \mathcal{R}_0) := \text{im}(I^q H^j(X, \mathcal{R}_0) \xrightarrow{i^*} I^p H^j(X, \mathcal{R}_0)) \quad (82)$$

and

$$I^{q \rightarrow p} \chi(X, \mathcal{R}_0) := \sum_{i=0}^n (-1)^i \dim(I^{q \rightarrow p} H^i(X, \mathcal{R}_0)) \quad (83)$$

Now we are ready to state the following proposition:

**Proposition 21.** *Let  $X$  be a compact and oriented smoothly stratified pseudomanifold of dimension  $n$  with a Thom-Mather stratification  $\mathfrak{X}$ . Let*

$$p : \{\text{Singular Strata of } X\} \rightarrow \mathbb{N}$$

*a general perversity such that*

$$p(Y) = -1$$

*for each stratum  $Y$  of  $X$  with  $\text{cod}(Y) = 1$ . Suppose that, if we call  $q$  its dual, then we have*

$$p \leq q.$$

*Then*

$$I^{q \rightarrow p} H^j(X, \mathcal{R}_0), \quad j = 0, \dots, n$$

*is a finite sequence of finite dimensional vector spaces with Poincaré duality. Analogously if*

$$p(Y) = 0$$

*for each stratum  $Y$  of  $X$  with  $\text{cod}(Y) = 1$  and*

$$p \geq q$$

*where  $q$  denote again the dual perversity of  $p$ , then*

$$I^{p \rightarrow q} H^j(X, \mathcal{R}_0), \quad j = 0, \dots, n$$

*is again a finite sequence of finite dimensional vector spaces with Poincaré duality.*

*Proof.* We know that  $p \leq q$ . This means that for each singular stratum of codimension  $i$  we have  $p \leq i - 2 - p$  that is  $p \leq \frac{i-2}{2}$ . But  $p$  takes values in  $\mathbb{N}$  and therefore  $p \leq \frac{i-2}{2}$  if and only if  $p \leq [\frac{i-2}{2}]$  that is  $p \leq \overline{m}$ . This implies that  $p$  satisfies the assumptions of theorem 14 that is there exist a quasi edge metric  $g$  on  $\text{reg}(X)$  such that  $p_g = p$ . In this way we can use proposition 20 to get the conclusion.

In the same way if  $p \geq q$  then we get  $p \geq \overline{m}$ . So we can use again theorem 14 and proposition 20 to get the assertion.  $\square$

We have the following four immediate corollaries:

**Corollary 10.** *In the hypothesis of proposition 21, if  $n$  is odd then*

$$I^{q \rightarrow p} \chi(X, \mathcal{R}_0) = 0 \quad (84)$$

*If  $n$  is even then*

$$I^{q \rightarrow p} \chi(X, \mathcal{R}_0) = \dim(I^{q \rightarrow p} H^{\frac{n}{2}}(X, \mathcal{R}_0)) \quad (85)$$

*when  $\frac{n}{2}$  is still even while if  $\frac{n}{2}$  is odd then*

$$I^{q \rightarrow p} \chi(X, \mathcal{R}_0) = -\dim(I^{q \rightarrow p} H^{\frac{n}{2}}(X, \mathcal{R}_0)) \quad (86)$$

**Corollary 11.** *In the same hypothesis of proposition 21 suppose that one of the two following properties is satisfied*

1.  $i_j^* : I^q H^j(X, \mathcal{R}_0) \longrightarrow I^p H^j(X, \mathcal{R}_0)$  is injective,
2.  $i_j^* : I^q H^j(X, \mathcal{R}_0) \longrightarrow I^p H^j(X, \mathcal{R}_0)$  is surjective.

Then

$$I^q H^j(X, \mathcal{R}_0), I^p H^j(X, \mathcal{R}_0) \quad j = 0, \dots, n \quad (87)$$

are a finite sequences of finite dimensional vector spaces with Poincaré duality.

**Corollary 12.** *In the hypothesis of proposition 21 we have the following inequalities:*

$$\dim(\text{im}(H_c^j(\text{reg}(X)) \xrightarrow{i^*} H_{dR}^j(\text{reg}(X)))) \leq \dim I^p H^j(X, \mathcal{R}_0) \quad (88)$$

$$\dim(\text{im}(H_c^j(\text{reg}(X)) \xrightarrow{i^*} H_{dR}^j(\text{reg}(X)))) \leq \dim I^q H^j(X, \mathcal{R}_0). \quad (89)$$

Moreover if on  $\text{reg}(X)$  we have that  $\text{im}(H_c^j(\text{reg}(X)) \xrightarrow{i_j^*} H_{dR}^j(\text{reg}(X)))$  is not trivial for some  $j$  then on  $X$   $I^p H^j(X, \mathcal{R}_0)$  and  $I^q H^j(X, \mathcal{R}_0)$  are always non trivial for each general per-versity  $p$  such that  $p \leq \underline{m}$  or  $p \geq \overline{m}$ . Finally, if on  $\text{reg}(X)$  we have that  $H_c^i(\text{reg}(X)) \rightarrow H_{dR}^i(\text{reg}(X))$  is injective, then we can improve the inequalities (88) and (89) in the following way:

$$\dim(H_c^j(\text{reg}(X))) \leq \dim I^p H^j(X, \mathcal{R}_0) \quad (90)$$

$$\dim(H_c^j(\text{reg}(X))) \leq \dim I^q H^j(X, \mathcal{R}_0) \quad (91)$$

$$b_{n-j}(\text{reg}(X)) \leq \dim I^p H^{n-j}(X, \mathcal{R}_0) \quad (92)$$

$$b_{n-j}(\text{reg}(X)) \leq \dim I^q H^{n-j}(X, \mathcal{R}_0) \quad (93)$$

*Proof.* All the previous inequalities from (88) to (91) are immediate consequences of the previous results. For the last two inequalities we observe that by Poincaré duality, we know that  $\dim(H_c^j(\text{reg}(X))) = \dim(H_{dR}^{n-j}(\text{reg}(X))) = b_{n-j}(\text{reg}(X))$ .

Moreover, from theorem 11, we know that  $H_{2,m \rightarrow M}^j(\text{reg}(X), g) \cong H_{2,m \rightarrow M}^{n-j}(\text{reg}(X), g)$ . Therefore using corollary 5 we get

$$b_{n-j}(\text{reg}(X)) \leq \dim(H_{2,m \rightarrow M}^{n-j}(\text{reg}(X), g)) \leq \dim(H_{2,\max}^{n-j}(\text{reg}(X), g)) = \dim(I^q H^{n-j}(X, \mathcal{R}_0))$$

$$b_{n-j}(\text{reg}(X)) \leq \dim(H_{2,m \rightarrow M}^{n-j}(\text{reg}(X), g)) \leq \dim(H_{2,\min}^{n-j}(\text{reg}(X), g)) = \dim(I^p H^{n-j}(X, \mathcal{R}_0))$$

and so the statement follows.  $\square$

Gluing together some of the previous results, now we can state the main result of this section. The first part is an Hodge theorem for  $\text{im}(I^{q_g} H^i(X, \mathcal{R}_0) \rightarrow I^{p_g} H^i(X, \mathcal{R}_0))$ , that is we will show the existence of a self-adjoint extension of  $\Delta_i : \Omega_c^i(\text{reg}(X)) \rightarrow \Omega_c^i(\text{reg}(X))$  having the nullspace isomorphic to  $\text{im}(I^{q_g} H^i(X, \mathcal{R}_0) \rightarrow I^{p_g} H^i(X, \mathcal{R}_0))$ . In the second part we will show that  $(d + \delta)_{ev}$ , that is the Gauss-Bonnet operator having as domain the space of the smooth forms of even degree with compact support, admits a Fredholm extension such that its index has a topological meaning.

**Theorem 15.** *In the same hypothesis of theorem 13; Let  $\Delta_{m,i}$  and  $(d_m + d_m^*)_{ev}$  the operators, as defined respectively in corollary 4 and proposition 12, associated to the riemannian manifold  $(\text{reg}(X), g)$ . Then we have the following results:*

$$\text{Ker}(\Delta_{m,i}) \cong \text{im}(I^{q_g} H^i(X, \mathcal{R}_0) \rightarrow I^{p_g} H^i(X, \mathcal{R}_0)) \quad (94)$$

$$\text{ind}((d_m + d_m^*)_{ev}) = I^{p_g \rightarrow q_g} \chi(X, \mathcal{R}_0). \quad (95)$$

*Proof.* (94) follows by theorem 13 and corollary 4; analogously (95) follows from theorem 13 and from proposition 12.  $\square$

Now suppose that  $\dim X = 4n$  where  $X$  is as in proposition 21. Let  $g$  be a quasi edge metric with weights on  $\text{reg}(X)$ . Then, by theorem 13, it follows that  $(L^2\Omega^i(\text{Reg}(X), g), d_{\max/\min, i})$  are Fredholm complexes and so  $(\text{Reg}(X), g)$  admits the  $L^2$  signature  $\sigma_2(\text{reg}(X), g)$  as defined in definition 6. Moreover, using again theorem 13, it follows that in this case the  $L^2$  signature  $\sigma_2(\text{reg}(X), g)$  is just the analytic version of the **perverse signature** introduced by Hunsicker in [14] in the case of  $\text{depth}(X) = 1$  and reintroduced in a purely topological way and generalized to any compact topological pseudomanifolds by Friedman and Hunsicker in [13]. In other words, if  $p_g$  is the general perversity of definition 11 and  $q_g$  it is its dual, then

$$\sigma_2(\text{reg}(X), g) = \sigma_{q_g \rightarrow p_g}(X) \quad (96)$$

and we provided an analytic way to construct  $\sigma_{q_g \rightarrow p_g}(X)$  when  $X$  is a smoothly stratified pseudomanifold with a Thom-Mather stratification which generalize the construction given by Hunsicker in [14] in the particular case of  $\text{depth}(X) = 1$ . (For the definition of  $\sigma_{q_g \rightarrow p_g}(X)$  see [13] pag. 15).

We have the following corollaries:

**Corollary 13.** *Let  $X$  as in theorem 13 and let  $g$  and  $h$  two quasi edge metrics with weights on  $\text{reg}(X)$ . If  $p_g = p_h$  then*

$$\sigma_2(\text{reg}(X), g) = \sigma_2(\text{reg}(X), h).$$

*Proof.* It follows immediately from theorem 13. □

**Corollary 14.** *Let  $X$  and  $X'$  as in theorem 13. Let  $g$  and  $h$  two a quasi edge metric with weights respectively on  $\text{reg}(X)$  and  $\text{reg}(X')$ . Let  $f : X \rightarrow X'$  a stratum preserving homotopy equivalence which preserves also the orientations of  $X$  and  $X'$ , see [19] pag 62 for the definition. Suppose that both  $p_g$  and  $p_h$  depend only on the codimension of the strata and that  $p_g = p_h$ . Then*

$$\sigma_2(\text{reg}(X), g) = \sigma_2(\text{reg}(X'), h).$$

*Proof.* As remarked above, by theorem 13, it follows that  $\sigma_2(\text{reg}(X), g)$  is the perverse signature of Friedman and Hunsicker associated to the general perversities  $p_g$  and  $t - p_g$ . Analogously  $\sigma_2(\text{reg}(X'), h)$  is the perverse signature of Friedman and Hunsicker associated to the general perversities  $p_h$  and  $t - p_h$ . So the statement follows by the invariance of the perverse signature under the action of stratum preserving homotopy equivalences which preserve also the orientations. □

### 3.3 Some examples of manifolds without riemannian metric with finite $L^2$ cohomology

Now we go ahead showing an example of a manifold  $M$  such that  $\text{im}(H_c^i(M) \rightarrow H_{dR}^i(M))$  is infinite dimensional. To do this we start with the following definition:

**Definition 12.** *Let  $M$  be a smooth manifold and let  $A \subset M$ .*

1. *We will say that  $A$  is bounded if its closure,  $\overline{A}$ , is compact.*
2. *We will say that  $M$  has only one end if for each compact subset  $K \subset M$   $M - K$  has only one unbounded connected component.*
3. *We will say that  $M$  has  $k$  ends (where  $k \geq 2$ ) if there is a compact set  $K_0 \subset M$  such that for every compact set  $K \subset M$  containing  $K_0$ ,  $M - K$  has exactly  $k$  unbounded connected components.*

The following proposition is a modified version of lemma 2.3 in [7]:

**Proposition 22.** *Let  $M$  a manifold with only one end. Then the natural map*

$$H_c^1(M) \rightarrow H_{dR}^1(M)$$

*is injective.*

*Proof.* Let  $\alpha \in \Omega_c^1(M)$  closed and let  $f : M \rightarrow \mathbb{R}$  a smooth function such that  $df = \alpha$ . This implies the existence of a constant  $c$  such that  $f|_{M-\text{supp}(\alpha)} = c$ . Therefore, by the fact that  $M$  has only one end, it follows that  $f - c$  has compact support.  $\square$

Now using Poincaré duality for open and oriented manifolds we know that the de Rham cohomology with compact support is infinite dimensional if and only if the de Rham cohomology is infinite dimensional. From this it follows that if  $M$  is a smooth and oriented manifold with only one end and such that  $H_{dR}^1(M)$  is infinite dimensional then also  $\text{im}(H_c^i(M) \rightarrow H_{dR}^i(M))$  is infinite dimensional. So we can state the following proposition:

**Proposition 23.** *Let  $M$  be an open and oriented surface with infinite genus and with only one end. Then  $\text{im}(H_c^i(M) \rightarrow H_{dR}^i(M))$  is infinite dimensional and therefore  $M$  does not admit a riemannian metric  $g$  (complete or incomplete) such that  $g$  implies one of the properties listed in the corollary 8.*

The rest of this subsection is devoted to show another example of an open manifold which satisfies corollary 8 but that it is not contemplated in the previous proposition. To do this we pass to state the following lemma which gives another sufficient condition to have  $\text{im}(H_c^i \rightarrow H_{dR}^i(M))$  infinite dimensional for some  $i$ .

**Lemma 1.** *Let  $M$  be an open and oriented smooth manifold of dimension  $n$ . Let  $\{A_j\}_{j \in J}$  a sequence of open subset such that*

1.  $\partial \overline{A_j}$  is smooth for each  $j$ .
2.  $\lim_{j \rightarrow \infty} \dim(\text{im}(H_c^i(A_j) \rightarrow H_{dR}^i(A_j))) = \infty$ .

*Then for the same  $i$   $\text{im}(H_c^i(M) \rightarrow H_{dR}^i(M))$  is infinite dimensional.*

*Proof.* It is an immediate consequence of the next proposition.  $\square$

**Proposition 24.** *Let  $M$  be an open and oriented smooth manifold of dimension  $n$ . Let  $A \subset M$  an open subset with smooth boundary. Then there is a natural injective maps*

$$\text{im}(H_c^{n-i}(A) \rightarrow H_{dR}^{n-i}(A)) \longrightarrow (\text{im}(H_c^i(M) \rightarrow H_{dR}^i(M)))^*$$

*Proof.* Consider the following pairing:

$$\text{im}(H_c^i(M) \rightarrow H_{dR}^i(M)) \times \text{im}(H_c^{n-i}(A) \rightarrow H_{dR}^{n-i}(A)) \longrightarrow \mathbb{R}, ([\eta], [\omega]) \mapsto \int_M \eta \wedge \omega \quad (97)$$

where  $\omega$  is a  $i$ -form closed with compact support in  $M$  and  $\eta$  is a closed  $(n-i)$ -form with compact support in  $A$ . As observed at the end of subsection 2.3 this pairing makes sense because a cohomology class in  $\text{im}(H_c^i(M) \rightarrow H_{dR}^i(M))$ , or in  $\text{im}(H_c^i(A) \rightarrow H_{dR}^i(A))$ , is just a cohomology class in  $H_{dR}^i(M)$ , or in  $H_{dR}^i(A)$ , such that it admits a representative with compact support respectively in  $M$  or  $A$ . Moreover from Poincaré duality for open and oriented manifold it follows immediately that this pairing is well defined. Now let  $[\omega] \in \text{im}(H_c^{n-i}(A) \rightarrow H_{dR}^{n-i}(A))$  such that for each class  $[\eta] \in \text{im}(H_c^i(M) \rightarrow H_{dR}^i(M))$  the pairing (97) is zero. This implies that for each smooth and closed  $i$  forms  $\phi$  with compact support in  $M$  we have

$$\int_M \phi \wedge \omega = 0.$$

In particular this is true for each smooth and closed  $i$  forms  $\phi$  with compact support in  $A$  and therefore, using again the Poincaré duality for open and oriented manifold, we get that there exists  $\beta \in \Omega^{n-i-1}(A)$  such that  $d\beta = \omega$ . So we can conclude that  $[\omega] = 0$  in  $\text{im}(H_c^i(A) \rightarrow H_{dR}^i(A))$  and this implies the statement.  $\square$

Using the previous lemma we have the following corollary that was suggested to the author by Pierre Albin:

**Corollary 15.** *Let  $M$  be an open and oriented surface obtained gluing an infinite but countable family of tori. Suppose that  $M$  has a finite number of ends. Then  $\text{im}(H_c^i(M) \rightarrow H_{dR}^i(M))$  is infinite dimensional and therefore  $M$  does not admit a riemannian metric  $g$  such that  $g$  implies one of the properties listed in the corollary 8.*

*Proof.* The idea is to show that this is a situation in which the previous lemma applies. By the assumptions for each  $j \in \mathbb{N}$  big enough, we can find an open subset  $A_j$  with the following properties:

1.  $M - A_j$  is disconnected, made of  $k$  unbounded components, where  $k$  is the number of ends of  $M$ .
2.  $\partial A_j$  is smooth, and made of  $k$  compact connected components.
3. By the compactness of  $\partial A_j$  it follows that each of its connected components is a compact smooth one dimensional manifold and therefore it is diffeomorphic to  $S^1$ . So we can glue to  $A_j$   $k$  copies of  $\overline{\mathbb{B}}$ , the unit ball in  $\mathbb{R}^2$  with boundary, to get a closed and oriented surfaces  $\Sigma_j$  of genus  $j$ .

Now, recalling that  $2 - 2j = \chi(\Sigma_j) = b_0(\Sigma_j) - b_1(\Sigma_j) + b_2(\Sigma_j)$  and using the Mayer-Vietoris sequence, it is not hard to see that  $\dim(H^1(A_j)) \geq 2j - k$  where  $k$  is the number of ends of  $M$  and therefore it is fixed. By the assumptions this implies that on  $M$  we can find a sequence of open subsets  $A_j$  such that

$$\lim_{j \rightarrow \infty} \dim(H_{dR}^1(A_j)) = \infty. \quad (98)$$

Now recall the fact that on a compact and oriented manifold with boundary  $\overline{M}$  we have  $H^i(\overline{M}, \partial\overline{M}) \cong H_c^i(M)$  where  $M$  is the interior of  $\overline{M}$ . So, from the long exact sequence for the relative de Rham cohomology on a compact manifold with boundary, it is easy to show that  $\dim(H^1(A_j)) = \dim(\text{im}(H_c^1(A_j) \rightarrow H_{dR}^1(A_j))) + \lambda_{A_j}$  where  $\lambda_{A_j} \in \{0, \dots, k\}$ . This means that the correction term  $\lambda_{A_j}$  could depends from  $A_j$  but in any case it lies in  $\{0, \dots, k\}$  which is a bounded set being  $k$  fixed. Therefore, from this equality and from (98), it follows that if we take a sequence of open subsets  $\{A_j\}$  such that each  $A_j$  satisfies the properties listed above then

$$\lim_{j \rightarrow \infty} \dim(\text{im}(H_c^1(A_j) \rightarrow H_{dR}^1(A_j))) = \infty.$$

This implies that we can apply lemma 1 and therefore the statement follows.  $\square$

Finally, using the notions introduced in definition 12 and proposition 22, we conclude the section giving another application to the stratified pseudomanifolds and intersection cohomology.

**Proposition 25.** *Let  $X$  be as in theorem 13. Suppose that  $X$  is normal, that is for each  $p \in \text{sing}(X)$  there exists an open neighbourhood  $U$  such that  $U - (U \cap \text{sing}(X))$  is connected. Then, if  $\text{sing}(X)$  is connected,  $\text{reg}(X)$  is an open manifold with only one end.*

*Proof.* Let  $K \subset \text{reg}(X)$  a compact subset. If  $\text{reg}(X) - K$  is connected then we have nothing to show. Suppose therefore that it is disconnected and let  $A_1, \dots, A_l$  the connected components. By the fact that  $X$  is normal it follows that there exists an open neighbourhood  $\text{sing}(X) \subset V \subset X$  such that  $V - \text{sing}(X)$  is connected. By the fact that  $K \subset \text{reg}(X)$  it follows that  $V = \bigcup_{i=1}^l (\overline{A_i} \cap V)$  and from this equality it follows that  $V - \text{sing}(X) = \bigcup_{i=1}^l (A_i \cap (V - \text{sing}(X)))$ . Every subset  $A_i \cap (V - \text{sing}(X))$  is an open subset of  $V - \text{sing}(X)$  and for each  $i, j \in \{1, \dots, l\}$  we have  $(A_i \cap (V - \text{sing}(X))) \cap (A_j \cap (V - \text{sing}(X))) = \emptyset$ . So the fact that  $V - \text{sing}(X)$  is connected, joined with the fact that  $V - \text{sing}(X) = \bigcup_{i=1}^l (A_i \cap (V - \text{sing}(X)))$ , implies that there exists just one index in  $\{1, \dots, l\}$ , which we label  $\gamma$ , such that  $A_\gamma \cap (V - \text{sing}(X)) \neq \emptyset$ . So we can conclude that

1.  $V - \text{sing}(X) \subset A_\gamma$ .
2.  $A_\gamma \cup \text{sing}(X)$  is open in  $X$ .

This implies that if we label  $\mathfrak{K}$  the closure in  $X$  of

$$\left( \bigcup_{i=1, i \neq \gamma}^l A_i \right) \cup K$$

then we have

$$\mathfrak{K} \subseteq X - (A_\gamma \cup \text{sing}(X)) \quad (99)$$

and therefore  $\mathfrak{K}$  is a compact subset of  $X$ . But from (99) it follows that  $\mathfrak{K} \subset \text{reg}(X)$  and therefore it is a compact subset of  $\text{reg}(X)$ . This allow us to conclude that for each  $i \in \{1, \dots, l\}$ ,  $i \neq \gamma$  we have that  $A_i$  is a compact subset of  $\text{reg}(X)$  and so we got the statement.  $\square$

We have the following corollary:

**Corollary 16.** *Let  $X$  be as in theorem 13 such that  $X$  is normal and  $\text{sing}(X)$  is connected. Let  $p$  be a general perversity as in the statement of theorem 14 and let  $q$  be its dual. Then we have the following inequalities:*

1.  $\dim(H_c^1(\text{reg}(X))) \leq I^p b_1(X, \mathcal{R}_0)$ ,  $\dim(H_c^1(\text{reg}(X))) \leq I^q b_1(X, \mathcal{R}_0)$ .
2.  $b_{n-1}(\text{reg}(X)) \leq I^p b_{n-1}(X, \mathcal{R}_0)$ ,  $b_{n-1}(\text{reg}(X)) \leq I^q b_{n-1}(X, \mathcal{R}_0)$ .

where  $I^p b_i(X, \mathcal{R}_0)$  is the dimension of  $I^p H^i(X, \mathcal{R}_0)$  and analogously  $I^q b_i(X, \mathcal{R}_0)$  is the dimension of  $I^q H^i(X, \mathcal{R}_0)$ . Finally if  $\dim X = 2$  and  $\text{cod}(\text{sing}(X)) = 0$  then

$$I^m \chi(X) \leq \chi(\text{reg}(X)) \quad (100)$$

where  $I^m \chi(X) = \sum_{i=0}^2 (-1)^i I^m b_i(X)$ .

*Proof.* From proposition 25 we know that  $\text{reg}(X)$  has only one end. Therefore from proposition 12 it follows that the maps  $H_c^1(M) \rightarrow H_{dR}^1(M)$  is injective and so the thesis follows by corollary 12. Before to prove the second part of the corollary we do the following observation: by the assumption it follows that  $H_c^1(\text{reg}(X))$  is finite dimensional; using Poincaré duality for open and oriented manifolds this implies that  $b_i(\text{reg}(X))$  is finite dimensional for each  $i = 0, \dots, 2$  and therefore  $\chi(\text{reg}(X))$  makes sense. Now by the assumptions on  $X$  it follows that  $\text{sing}(X) = \{p\}$  and  $X$  is a Witt space (For the definition of Witt space see for example [19] pag 75). It is well known that, over a Witt space, the intersection cohomology associated to the lower middle perversity satisfies has the Poincaré duality, that is we have  $I^m H^i(X) \cong I^m H^{2-i}(X)$ . Poincaré duality for open and oriented manifolds implies that  $b_2(\text{reg}(X)) = \dim(H_c^0(\text{reg}(X))) = 0$ . So, using the previous statements of this corollary, we have  $I^m \chi(X) = -I^m b_1(X) \leq -b_1(\text{reg}(X)) \leq 1 - b_1(\text{reg}(X)) = \chi(\text{reg}(X))$ .  $\square$

### 3.4 Some applications to the Friedrichs extension of $\Delta_i$

This last section is devote to show some properties of the Friedrichs extension of  $\Delta_i : \Omega_c^i(M) \rightarrow \Omega_c^i(M)$ .

The main result is to show that if  $(M, g)$  is an open and oriented riemannian manifold such that  $(L^2 \Omega^*(M, g), d_{\max/\min, *})$  are Fredholm complexes then, for each  $i = 0, \dots, \dim M$ , the Friedrichs extension of  $\Delta_i : \Omega_c^i(M) \rightarrow \Omega_c^i(M)$  is a Fredholm operator. In particular this applies when  $M$  is the regular part of a compact and smoothly stratified pseudomanifold with a Thom-Mather stratification and  $g$  is a quasi edge metric with weights on  $\text{reg}(X)$ . We start recalling the definition of the Friedrichs extension:

**Definition 13.** *Let  $H$  be an Hilbert space and  $B : H \rightarrow H$  a densely defined operator. Suppose that  $B$  is positive, that is for each  $u \in \mathcal{D}(B)$  we have  $\langle Bu, u \rangle \geq 0$ . The Friedrichs extension of  $B$ , usually labeled  $B^{\mathcal{F}}$ , is the operator defined in the following way:*

$$\begin{aligned} \mathcal{D}(B^{\mathcal{F}}) = \{ & u \in \mathcal{D}(B^*) : \text{there exists } \{u_n\} \subset \mathcal{D}(B) \text{ such that } \langle u - u_n, u - u_n \rangle \rightarrow 0 \text{ and} \\ & \langle B(u_n - u_m), u_n - u_m \rangle \rightarrow 0 \text{ for } n, m \rightarrow \infty \} \text{ and we put } B^{\mathcal{F}}(u) = B^*(u). \end{aligned}$$

**Proposition 26.** *In the same assumptions of the previous definition  $B^{\mathcal{F}}$  is a positive self-adjoint extension of  $B$ .*

*Proof.* See [20] appendix C.  $\square$

**Lemma 2.** *Let  $A_j : H_j \rightarrow H_j$ ,  $j = 1, 2$ , two positive and densely defined operators. Then on  $H_1 \oplus H_2$ , with the natural Hilbert space structure of a direct sum, we have*

$$(A_1 \oplus A_2)^{\mathcal{F}} = A_1^{\mathcal{F}} \oplus A_2^{\mathcal{F}}.$$

*Proof.* It follows from the assumptions of the lemma that  $A_1 \oplus A_1 : H_1 \oplus H_2 \rightarrow H_1 \oplus H_2$  is densely defined and positive. Moreover it clear that  $(A_1 \oplus A_2)^* = A_1^* \oplus A_2^*$ . Now let  $(a, b) \in \mathcal{D}((A_1 \oplus A_2)^\mathcal{F})$ . From definition 13 it follows that  $(a, b) \in \mathcal{D}((A_1 \oplus A_2)^*)$  and there exists a sequence  $\{(a_n, b_n)\} \subset \mathcal{D}(A_1 \oplus A_2)$  such that:

$$(a_n, b_n) \rightarrow (a, b) \text{ and } \langle A \oplus B((a_n, b_n) - (a_m, b_m)), (a_n, b_n) - (a_m, b_m) \rangle \rightarrow 0.$$

Furthermore from the same definition we know that  $(A_1 \oplus A_2)^\mathcal{F}(a, b) = (A_1 \oplus A_2)^*(a, b)$ . But from these requirements it follows immediately that  $a \in \mathcal{D}(A_1^*)$ ,  $b \in \mathcal{D}(A_2^*)$ ,  $\{a_n\} \subset \mathcal{D}(A_1)$ ,  $\{b_n\} \subset \mathcal{D}(A_2)$ ,  $a_n \rightarrow a$ ,  $\langle A_1(a_n - a_m), a_n - a_m \rangle \rightarrow 0$  and analogously that  $b_n \rightarrow b$  and that  $\langle A_2(b_n - b_m), b_n - b_m \rangle \rightarrow 0$ . So it follows that  $a \in \mathcal{D}(A_1^\mathcal{F})$ ,  $b \in \mathcal{D}(A_2^\mathcal{F})$  and  $(A_1 \oplus A_2)^\mathcal{F}(a, b) = A_1^\mathcal{F}(a) \oplus A_2^\mathcal{F}(b)$ . In this way we know that  $A_1^\mathcal{F} \oplus A_2^\mathcal{F}$  is an extension of  $(A_1 \oplus A_2)^\mathcal{F}$ . Moreover it is clear that also  $A_1^\mathcal{F} \oplus A_2^\mathcal{F}$  it is a self-adjoint operator because it is a direct sum of two self-adjoint operators acting on  $H_1$  and  $H_2$  respectively. Finally, by the fact that both  $A_1^\mathcal{F} \oplus A_2^\mathcal{F}$  and  $(A_1 \oplus A_2)^\mathcal{F}$  are self-adjoint operators, it follows that  $A_1^\mathcal{F} \oplus A_2^\mathcal{F} = (A_1 \oplus A_2)^\mathcal{F}$ .  $\square$

**Remark 6.** It clear that the previous proposition generalizes to the case of a finite sum, that is if we have  $A_j : H_j \rightarrow H_j$   $j = 1, \dots, n$  such that for each  $j$   $A_j$  is positive and densely defined then:

$$(A_1 \oplus \dots \oplus A_n)^\mathcal{F} : \bigoplus_{j=1}^n H_j \rightarrow \bigoplus_{j=1}^n H_j = A_1^\mathcal{F} \oplus \dots \oplus A_n^\mathcal{F} : \bigoplus_{j=1}^n H_j \rightarrow \bigoplus_{j=1}^n H_j \quad (101)$$

**Lemma 3.** Let  $E, F$  be two vector bundles over an open, incomplete and oriented riemannian manifold  $(M, g)$ . Let  $g$  and  $h$  be two metrics on  $E$  and  $F$  respectively. Let  $d : C_c^\infty(M, E) \rightarrow C_c^\infty(M, F)$  an unbounded and densely defined differential operator. Let  $d^t : C_c^\infty(M, F) \rightarrow C_c^\infty(M, E)$  its formal adjoint. Then for  $d^t \circ d : L^2(M, E) \rightarrow L^2(M, E)$  we have:

$$(d^t \circ d)^\mathcal{F} = d_{\max} \circ d_{\min}.$$

*Proof.* See [6], lemma 3.1 pag. 447.  $\square$

From lemma 3 we get, as it is showed in [6] pag. 448, the following useful corollary:

**Corollary 17.** Let  $(M, g)$  be an open, oriented and incomplete riemannian manifold of dimension  $n$ . Consider the Laplacian acting on the space of smooth forms with compact support:

$$\Delta : \bigoplus_{i=0}^n \Omega_c^i(M) \longrightarrow \bigoplus_{i=0}^n \Omega_c^i(M).$$

Then for

$$\Delta^\mathcal{F} : \bigoplus_{i=0}^n L^2 \Omega^i(M, g) \longrightarrow \bigoplus_{i=0}^n L^2 \Omega^i(M, g)$$

we have

$$\Delta^\mathcal{F} = (d + \delta)_{\max} \circ (d + \delta)_{\min}.$$

Now we are in positions to state the following result:

**Theorem 16.** Let  $(M, g)$  be an open, oriented and incomplete riemannian manifold of dimension  $n$ . Then for each  $i = 0, \dots, n$  we have the following properties:

1.  $\text{Ker}(\Delta_i^\mathcal{F}) = \mathcal{H}_{\min}^i(M, g) = \text{Ker}(\Delta_{\min, i})$ ,  $\overline{\text{ran}(\Delta_i^\mathcal{F})} = \overline{\text{ran}(\Delta_{\max, i})}$ .
2. If  $\overline{H}_{2, m \rightarrow M}^i(M, g)$  is finite dimensional then  $\text{Ker}(\Delta_i^\mathcal{F})$  is finite dimensional.
3. If  $(L^2 \Omega^*(M, g), d_{\max, *})$  is a Fredholm complex, or equivalently if  $(L^2 \Omega^*(M, g), d_{\min, *})$  is a Fredholm complex, then for each  $i$   $\Delta_i^\mathcal{F}$  is a Fredholm operator on its domain endowed with graph norm and  $\text{ran}(\Delta_i^\mathcal{F}) = \text{ran}(\Delta_{\max, i})$ .



*Proof.* In the first point the equality  $\mathcal{H}_{min}^i(M, g) = Ker(\Delta_{min,i})$  is showed in [4] prop. 5. For the other equality, from lemma 2 and corollary 17, we know that

$$(d + \delta)_{max} \circ (d + \delta)_{min} = \Delta^{\mathcal{F}} = \oplus_{i=0}^n \Delta_i^{\mathcal{F}}$$

and therefore

$$Ker((d + \delta)_{max} \circ (d + \delta)_{min}) = Ker(\oplus_{i=0}^n \Delta_i^{\mathcal{F}}) = \oplus_{i=0}^n Ker(\Delta_i^{\mathcal{F}}).$$

But for  $Ker((d + \delta)_{max} \circ (d + \delta)_{min})$  we have  $Ker((d + \delta)_{max} \circ (d + \delta)_{min}) = Ker(d + \delta)_{min} = Ker(d_{min}) \cap Ker(\delta_{min}) = \oplus_{i=0}^n Ker(d_{min,i}) \cap Ker(\delta_{min,i-1})$ . The first equality follows by the fact that for each  $\eta \in \mathcal{D}((d + \delta)_{max} \circ (d + \delta)_{min})$  we have  $\langle (d + \delta)_{max} \circ (d + \delta)_{min}(\eta), \eta \rangle_{L^2\Omega(M,g)} = \langle (d + \delta)_{min}(\eta), (d + \delta)_{min}(\eta) \rangle_{L^2\Omega(M,g)}$ .

For the second equality it is clear that  $Ker(d + \delta)_{min} \subseteq Ker(d_{min}) \cap Ker(\delta_{min})$ . But  $Ker(d_{min}) \cap Ker(\delta_{min}) = \overline{ran(d_{max}) + ran(\delta_{max})}^\perp$  and  $Ker(d + \delta)_{min} = \overline{ran((d + \delta)_{max})}^\perp$ . By the fact that  $\overline{ran((d + \delta)_{max})} \subseteq \overline{ran(d_{max}) + ran(\delta_{max})}$  it follows that  $Ker(d_{min}) \cap Ker(\delta_{min}) \subseteq Ker((d + \delta)_{min})$  and so we have obtained the second equality. The last equality follows because  $\overline{ran(d_{max}) + ran(\delta_{max})} = \oplus_{i=0}^n \overline{ran(d_{max,i-1}) + ran(\delta_{max,i})}$  that is

$$(Ker(d_{min}) \cap Ker(\delta_{min}))^\perp = \oplus_{i=0}^n (Ker(d_{min,i}) \cap Ker(\delta_{min,i-1}))^\perp$$

and both  $Ker(d_{min}) \cap Ker(\delta_{min})$  and  $\oplus_{i=0}^n Ker(d_{min,i}) \cap Ker(\delta_{min,i-1})$  are closed. In this way can conclude that  $\oplus_{i=0}^n Ker(\Delta_i^{\mathcal{F}}) = \oplus_{i=0}^n \mathcal{H}_{min}^i(M, g)$  and therefore that

$$\mathcal{H}_{min}^i(M, g) = Ker(\Delta_i^{\mathcal{F}}).$$

Finally, using the fact that  $\Delta_i^{\mathcal{F}}$  is self-adjoint and that  $\Delta_{min,i} = (\Delta_{max,i})^*$  it follows that

$$\overline{ran(\Delta_i^{\mathcal{F}})} = \overline{ran(\Delta_{max,i})}.$$

For the second point, if we call  $\pi_{abs,i} : L^2\Omega^i(M, g) \rightarrow \mathcal{H}_{abs}^i(M, g)$  the projection on  $\mathcal{H}_{abs}^i(M, g)$ , we know that  $\pi_{abs,i}(\mathcal{H}_{rel}^i) \cong \overline{H}_{2,m \rightarrow M}^i(M, g)$ . This property is showed in a more general context in the proof of theorem 7 and remarked in remark 1. But  $\mathcal{H}_{min}^i(M, g) = Ker(d_{min,i}) \cap Ker(\delta_{min,i-1}) = \mathcal{H}_{abs}^i(M, g) \cap \mathcal{H}_{rel}^i(M, g)$ . So  $\mathcal{H}_{min}^i(M, g) \subseteq \pi_{abs,i}(\mathcal{H}_{rel}^i(M, g))$  and therefore the second statement follows.

Now consider the third point; we want to show that if  $(L^2\Omega^*(M, g), d_{max,*})$  is a Fredholm complex then also  $(d + \delta)_{max} \circ (d + \delta)_{min} : \oplus_{i=0}^n L^2\Omega^i(M, g) \rightarrow \oplus_{i=0}^n L^2\Omega^i(M, g)$  is a Fredholm operator. By the previous point, we already know that the nullspace of  $(d + \delta)_{max} \circ (d + \delta)_{min}$  is finite dimensional. So we have to show that its range is closed with finite dimensional orthogonal complement. To do this is equivalent to show that the cokernel of  $(d + \delta)_{max} \circ (d + \delta)_{min}$  is finite dimensional. We will do this showing that  $ran((d + \delta)_{max} \circ (d + \delta)_{min}) = ran((d + \delta)_{max})$  and that  $(d + \delta)_{max}$  has finite dimensional cokernel. To do this we observe that, by the fact that  $(d + \delta)_{min}^* = (d + \delta)_{max}$ , it follows that

$$ran((d + \delta)_{max}) = \{(d + \delta)_{max}(u) : u \in \overline{ran((d + \delta)_{min})} \cap \mathcal{D}((d + \delta)_{max})\}. \quad (102)$$

Now, as we showed in corollary 6 of [4], if  $(L^2\Omega^*(M, g), d_{max,*})$  is a Fredholm complex then  $d_{max} + \delta_{min}$  is a Fredholm operator. But the fact that  $ran(d_{max} + \delta_{min}) \subset ran((d + \delta)_{max})$  implies that there is a surjective map

$$\frac{(\oplus_{i=0}^n L^2\Omega^i(M, g))}{ran((d + \delta)_{max})} \longrightarrow \frac{(\oplus_{i=0}^n L^2\Omega^i(M, g))}{ran(d_{max} + \delta_{min})}.$$

So  $(d + \delta)_{max}$  on its domain with the graph norm is a bounded linear operator with finite dimensional cokernel and this implies that the range of  $(d + \delta)_{max}$  is closed with finite dimensional orthogonal complement. But  $((d + \delta)_{max})^* = (d + \delta)_{min}$  and therefore also  $(d + \delta)_{min}$  has closed range. In this way (102) becomes

$$ran((d + \delta)_{max}) = \{(d + \delta)_{max}(u) : u \in ran((d + \delta)_{min}) \cap \mathcal{D}((d + \delta)_{max})\}. \quad (103)$$

So we can conclude that  $ran((d + \delta)_{max} \circ (d + \delta)_{min}) = ran((d + \delta)_{max})$  and therefore  $(d + \delta)_{max} \circ (d + \delta)_{min}$  is a Fredholm operator.

Now, by the equality  $(d + \delta)_{max} \circ (d + \delta)_{min} = \oplus_{i=0}^n \Delta_i^{\mathcal{F}}$ , we get, for each  $i = 0, \dots, n$ , that also  $\Delta_i^{\mathcal{F}}$  has closed range. Moreover we already know that its nullspace of  $\Delta_i^{\mathcal{F}}$  is finite dimensional and so, because it is self-adjoint and with closed range, we can conclude that it is Fredholm. Finally, as we showed in [4] corollary 6, we know that  $\Delta_{max,i}$  has finite dimensional cokernel and so we can conclude that  $\text{ran}(\Delta_{max,i}) = \text{ran}(\Delta_i^{\mathcal{F}})$ .  $\square$

As mentioned at the beginning of the section the following corollary is an application of the previous theorem; it already known when  $X$  is a compact manifold with isolated singularities for any positive conic operator (see [18]) and also for  $\Delta_i^{\mathcal{F}}$  when  $(M, g)$  is a manifold with incomplete edges, see [21].

**Corollary 18.** *Let  $X$  be a compact smoothly and oriented stratified pseudomanifold of dimension  $n$  with a Thom Mather stratification. Let  $g$  be a quasi-edge metric with weights on  $\text{reg}(X)$ . Then on  $L^2\Omega^i(\text{reg}(X), g)$ , for each  $i = 0, \dots, n$ ,  $\Delta_i^{\mathcal{F}}$  is a Fredholm operator; moreover  $\text{ran}(\Delta_i^{\mathcal{F}}) = \text{ran}(\Delta_{max,i})$  and  $\text{Ker}(\Delta_i^{\mathcal{F}}) = \text{Ker}(\Delta_{min,i}) = \mathcal{H}_{min}^i(M, g)$ .*

## 4 Final considerations

Consider again an open, oriented and incomplete riemannian manifold  $(M, g)$  of dimension  $n$ . By corollary 5 we now that that there is a copy of  $\text{im}(\overline{H}_{2,min}^i(M, g) \rightarrow \overline{H}_{2,max}^i(M, g))$  in each  $i$ -th reduced cohomology group  $\overline{H}_{2,D_*}^i(M, g)$  of each closed extension  $(L^2\Omega^*(M, g), D_*)$  of  $(\Omega_c^*(M), d_*)$ . In the same way, using again corollary 5, we know that there is a copy of  $\text{im}(H_{2,min}^i(M, g) \rightarrow H_{2,max}^i(M, g))$  in each  $i$ -th reduced cohomology group  $H_{2,D_*}^i(M, g)$  of each closed extension  $(L^2\Omega^*(M, g), D_*)$  of  $(\Omega_c^*(M), d_*)$ . In particular, by theorem 12, we know that when  $d_{min,i}$  has closed range for each  $i$  then the groups  $\text{im}(H_{2,min}^i(M, g) \rightarrow H_{2,max}^i(M, g))$  are really the cohomology groups of an Hilbert complex that we labeled  $(L^2\Omega^i(M, g), d_{m,i})$ . Therefore we can look at  $\text{im}(H_{2,min}^i(M, g) \rightarrow H_{2,max}^i(M, g))$  as the smallest possible  $L^2$ -cohomology groups for  $(M, g)$ .

From the Hodge point of view the smallest Hodge cohomology groups are  $\mathcal{H}_{min}^i(M, g)$  defined, for each  $i = 0, \dots, n$ , as  $\text{Ker}(d_{min,i}) \cap \text{Ker}(\delta_{min,i-1})$  or equivalently, see [4] proposition 5, as the nullspace of  $\Delta_{min,i}$ , where  $\Delta_{min,i}$  is the minimal closed extension of  $\Delta_i : \Omega_c^i(M) \rightarrow \Omega_c^i(M)$ . Therefore a natural question is:

- Is there any relations between  $\mathcal{H}_{min}^i(M, g)$  and  $\text{im}(\overline{H}_{2,min}^i(M, g) \rightarrow \overline{H}_{2,max}^i(M, g))$  or between  $\mathcal{H}_{min}^i(M, g)$  and  $\text{im}(H_{2,min}^i(M, g) \rightarrow H_{2,max}^i(M, g))$ ?

In [15] theorem 4.8, using techniques arising from Mazzeo's edge calculus, the author showed that if  $(M, g)$  is an incomplete manifold with edge then we have the following isomorphism:

$$\mathcal{H}_{min}^i(M, g) \cong \text{im}(H_{2,min}^i(M, g) \rightarrow H_{2,max}^i(M, g)). \quad (104)$$

Therefore, using corollary 4 and theorem 16, we get the following immediate consequences:

**Corollary 19.** *Let  $(M, g)$  be an incomplete manifold with edge. Then, for each  $i = 0, \dots, n$*

1.  $\text{Ker}(\Delta_{m,i}) = \text{Ker}(\Delta_{min,i}) = \text{Ker}(\Delta_i^{\mathcal{F}})$
2.  $\text{ran}(\Delta_{m,i}) = \text{ran}(\Delta_{max,i}) = \text{ran}(\Delta_i^{\mathcal{F}})$

Finally we conclude the section showing that the isomorphism (104) is equivalent to require that the Hilbert space  $L^2\Omega^i(M, g)$  satisfies some geometric properties.

**Proposition 27.** *Let  $(M, g)$  an open oriented and incomplete riemannian manifold. Suppose that, for each  $i = 0, \dots, n$ ,  $\text{im}(\overline{H}_{2,min}^i(M, g) \rightarrow \overline{H}_{2,max}^i(M, g))$  is finite dimensional. Then there exists always an injective map*

$$\mathcal{H}_{min}^i(M, g) \rightarrow \text{im}(\overline{H}_{2,min}^i(M, g) \rightarrow \overline{H}_{2,max}^i(M, g)).$$

Moreover the following properties are equivalent:

1.  $\mathcal{H}_{min}^i(M, g) \cong \text{im}(\overline{H}_{2,min}^i(M, g) \rightarrow \overline{H}_{2,max}^i(M, g))$

2.  $\mathcal{H}_{abs}^i(M, g) = \mathcal{H}_{min}^i(M, g) \oplus (\overline{ran(\delta_{max,i})} \cap \mathcal{H}_{abs}^i(M, g))$
3. Let  $\pi_{abs/rel/min,i} : L^2\Omega^i(M, g) \rightarrow \mathcal{H}_{abs/rel/min}^i(M, g)$  be the orthogonal projection of  $L^2\Omega^i(M, g)$  respectively on  $\mathcal{H}_{abs}^i(M, g)$ ,  $\mathcal{H}_{rel}^i(M, g)$  and  $\mathcal{H}_{min}^i(M, g)$ .  
Then  $\pi_{rel,i} \circ \pi_{abs,i} = \pi_{min,i} = \pi_{abs,i} \circ \pi_{rel,i}$ .
4.  $\mathcal{H}_{rel}^i(M, g) = \mathcal{H}_{min}^i(M, g) \oplus (\overline{ran(d_{max,i})} \cap \mathcal{H}_{rel}^i(M, g))$
5.  $\overline{ran(d_{max,i})} = (\overline{ran(d_{max,i})} \cap \mathcal{H}_{rel}^i(M, g)) \oplus \overline{ran(d_{min,i})} \oplus (\overline{ran(d_{max,i})} \cap \overline{ran(\delta_{max,i})})$

Finally, if  $(L^2\Omega^i(M, g), d_{max,i})$  or equivalently  $(L^2\Omega^i(M, g), d_{min,i})$  is a Fredholm complex then there exists always an injective map

$$\mathcal{H}_{min}^i(M, g) \rightarrow \text{im}(H_{2,min}^i(M, g) \rightarrow H_{2,max}^i(M, g)).$$

Moreover the previous four equivalent conditions become:

1.  $\mathcal{H}_{min}^i(M, g) \cong \text{im}(H_{2,min}^i(M, g) \rightarrow H_{2,max}^i(M, g))$
2.  $\mathcal{H}_{abs}^i(M, g) = \mathcal{H}_{min}^i(M, g) \oplus (ran(\delta_{max,i}) \cap \mathcal{H}_{abs}^i(M, g))$
3. Let  $\pi_{abs/rel/min,i} : L^2\Omega^i(M, g) \rightarrow \mathcal{H}_{abs/rel/min}^i(M, g)$  be the orthogonal projection of  $L^2\Omega^i(M, g)$  respectively on  $\mathcal{H}_{abs}^i(M, g)$ ,  $\mathcal{H}_{rel}^i(M, g)$  and  $\mathcal{H}_{min}^i(M, g)$ .  
Then  $\pi_{rel,i} \circ \pi_{abs,i} = \pi_{min,i} = \pi_{abs,i} \circ \pi_{rel,i}$ .
4.  $\mathcal{H}_{rel}^i(M, g) = \mathcal{H}_{min}^i(M, g) \oplus (ran(d_{max,i}) \cap \mathcal{H}_{rel}^i(M, g))$
5.  $ran(d_{max,i}) = (ran(d_{max,i}) \cap \mathcal{H}_{rel}^i(M, g)) \oplus ran(d_{min,i}) \oplus (ran(d_{max,i}) \cap ran(\delta_{max,i}))$

*Proof.* Clearly it is enough to prove just the first part of the proposition. The second part follows by the first part of the proposition and by the fact that if  $(L^2\Omega^i(M, g), d_{max/min,i})$  is a Fredholm complex then  $d_{max/min,i}$  has closed range. Let  $\pi_{1,i} : \mathcal{H}_{rel}^i(M, g) \rightarrow \mathcal{H}_{abs}^i(M, g)$ ,  $\pi_{4,i} : \mathcal{H}_{abs}^i(M, g) \rightarrow \mathcal{H}_{rel}^i(M, g)$  as defined in the proof of theorem 7. Moreover, by proposition 8, we know that  $(\pi_{1,i})^* = \pi_{4,i}$  and analogously  $(\pi_{1,i})^* = \pi_{4,i}$ . By the proof of theorem 7 we know that  $\pi_{1,i}(\mathcal{H}_{rel}^i(M, g)) \cong \text{im}(\overline{H}_{2,min}^i(M, g) \rightarrow \overline{H}_{2,max}^i(M, g))$ . Clearly, by the fact that  $\mathcal{H}_{min}^i(M, g) = \mathcal{H}_{abs}^i(M, g) \cap \mathcal{H}_{rel}^i(M, g)$ , it follows that  $\mathcal{H}_{min}^i(M, g) \subset \pi_{1,i}(\mathcal{H}_{rel}^i(M, g))$  and so we got the first assertion.

Now we pass to show that 1)  $\Rightarrow$  2). As recalled above we know that  $\pi_{1,i}(\mathcal{H}_{rel}^i(M, g)) \cong \text{im}(\overline{H}_{2,min}^i(M, g) \rightarrow \overline{H}_{2,max}^i(M, g))$  and that  $\mathcal{H}_{min}^i(M, g) = \mathcal{H}_{abs}^i(M, g) \cap \mathcal{H}_{rel}^i(M, g)$ ; therefore using 1) it follows that  $\mathcal{H}_{min}^i(M, g) = \pi_{1,i}(\mathcal{H}_{rel}^i(M, g))$ . This implies that  $(\mathcal{H}_{min}^i(M, g))^\perp \cap \mathcal{H}_{abs}^i(M, g) = (\pi_{1,i}(\mathcal{H}_{rel}^i(M, g)))^\perp \cap \mathcal{H}_{abs}^i(M, g) = \text{Ker}(\pi_{4,i}) = \overline{ran(\delta_{max,i})} \cap \mathcal{H}_{abs}^i(M, g)$  and this complete the proof of the first implication.

Now suppose that 2) is satisfied. Then it is immediate that  $\pi_{rel,i} \circ \pi_{abs,i} = \pi_{min,i}$  and therefore it is an easy consequence that also  $\pi_{abs,i} \circ \pi_{rel,i} = \pi_{min,i}$ . Moreover it is still immediate that 3)  $\Rightarrow$  4) because in this case  $\pi_{4,i}(\mathcal{H}_{abs}^i(M, g)) = \mathcal{H}_{min}^i(M, g)$ . Now we want to show that 4)  $\Rightarrow$  5). Clearly  $\mathcal{H}_{min}^i(M, g)$  is orthogonal to  $\overline{ran(\delta_{max,i})}$  and to  $\overline{ran(d_{max,i})}$ . This implies that the range of the orthogonal projection of  $\overline{ran(d_{max,i})}$  onto  $\mathcal{H}_{rel}^i(M, g)$  is just the intersection  $\mathcal{H}_{rel}^i(M, g) \cap \overline{ran(d_{max,i})}$ . From this it follows that also the range of the orthogonal projection of  $\overline{ran(d_{max,i})}$  onto  $\overline{ran(\delta_{max,i})}$  is just the intersection  $\overline{ran(d_{max,i})} \cap \overline{ran(\delta_{max,i})}$  and therefore the implication 3)  $\Rightarrow$  4) is proved. Finally, if 5) holds, it is immediate to show that  $\pi_{1,i}(\mathcal{H}_{rel}^i(M, g)) = \mathcal{H}_{min}^i(M, g)$  and this, using the fact that  $\pi_{1,i}(\mathcal{H}_{rel}^i(M, g)) \cong \text{im}(\overline{H}_{2,min}^i(M, g) \rightarrow \overline{H}_{2,max}^i(M, g))$  implies 1). This complete the proof of the proposition.  $\square$

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Dipartimento di Matematica, Sapienza Università di Roma  
 E-mail address: bei@math.uniroma1.it